A framework for understanding what Algebraic Thinking is

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On this paper a framework (the **Numerical-Analogical** framework) is proposed in order to provide a reference for investigations (both theoretical and experimental) on the nature of Algebraic Thinking. The framework is described and its adequacy is demonstrated by examining: experimental evidence from students' work (both new and previous findings), the historical development of algebra and algebra as a subject-matter in Mathematics. A characterisation of Algebraic Thinking on the basis of the Numerical-Analogical framework is provided. The belief that Algebraic Thinking can only happen in the context of algebraic symbolism is shown to be erroneous and misleading.

> *"But neither of them was able to prove the theorem, and Waring confessed that the demonstration seemed more difficult because no* notation *can be devised to express a prime number. But in our opinion truths of this kind should be drawn from notions rather than from notations"*
>
> C.F. Gauss, on Wilson's theorem, in *Disquisitiones Arithmeticae*

1. Introduction

Until now, a substantial amount of information has been gathered on the learning of school algebra (eg, Collis,1982;Küchemann,1984; Wheeler & Lee,1987; Bell,1987), but nevertheless, a clear characterisation for "Algebraic Thinking" is still missing (Kieran,1989; Lee,1987).

As a whole, that research has been strongly focused on investigating Algebraic Thinking as the mode of thinking that goes with "doing algebra" (either interpreting or manipulating algebraic statements or using algebra to solve problems and explore situations), rather than the mode of thinking that allows the development of algebra.. A consequence of this "content-driven" approach is that the students' "informal" solutions have been characterised more in terms of misinterpretations and failure to "understand" and less in terms of what they are **actually** doing .

L. Booth suggested that the sources of those misunderstandings (or lack of understanding, as it might be more adequate) are to be found in an incompatibility between the "informal" methods used by the students and the methods of algebra rather than in developmental obstacles (in the sense of Piaget) (see, for example, Booth, 1984). We strongly share this point of view, and investigating the nature of those "informal" solutions, at the same time we investigate the **nature** of "algebraic" solutions, has been the central objective of a set of studies carried out by the author for the last two years, aiming at identifying possible source(s) for that incompatibility.

A framework that helps us to understand the twofold nature of this question, is one that enables us to handle the different meanings that can be attached to the elements involved in the situation that is being dealt with by the students: numbers, operations and arithmetical and algebraic symbolism (where they are involved), but also the imagery suggested or provided by the situation or used as a support for reasoning (the context of "realistic" problems, diagrams, etc). In speaking of "meaning" we are inevitably led to **referentials**, and this is what our framework has to provide in the first place: a description of **different fields of reference** in which **different interpretations** of those elements produce **solutions of different nature**.

This paper is a result of the work being carried out by the author as part of his PhD studies, under the supervision of Dr Alan W Bell, at Nottingham University.

A first important consequence of thinking in terms of distinct fields of reference within which the elements of a situation are interpreted, is that our approach is not content-driven: the same framework can be applied to the analysis of solutions of "realistic" and "purely numerical" problems, problems set in algebraic language and "verbal" problems. Also of considerable importance, such framework can be applied to the analysis of the algebra of the "ancients", and this might shed some new light onto a possible paralellism between the historical development of algebra and the aqcisition of algebraic thinking by individuals.

In the next four sections such a framework is sketched and support for its adequacy is drawn from three sources: the historical development of algebra, algebra as a theoretical discipline and empirical evidence from investigations on students' solutions. On the last section we return to the framework and its characteristics are fully described.

2.THE NUMERICAL-ANALOGICAL FRAMEWORK

Our framework distinguishes between two basic fields of reference: the **Numerical field of reference** and the **Analogical field of reference**.

To operate within the **Numerical field of reference** means that only the "arithmetical" environment is relevant to the process of manipulating or exploring a situation. If it is the case of solving a problem, the problem is solved through the manipulation of the numerical relationships contained in or described or allowed by it, and this process is guided by the arithmetical structure of those relationships and by the principles that are recognized as governing the arithmetical environment.

To operate within the **Analogical field of reference** means that a situation is manipulated or explored by manipulating features of the situation itself. Arithmetical operations are used to **evaluate** parts, and the choice of operation to be used is made on the basis of a **qualitative** analysis of the situation or problem that is being examined.

The framework we propose here has two fundamental characteristics:

(i) it rejects the idea of a "**pre**-algebraic" mode of thinking, something that when extended or further developed leads to an "algebraic" mode of thinking; we use instead the idea of a "**non**-algebraic" mode of thinking; the "meaninglessness" pointed out by students is interpreted not in terms of the "meaninglessness" of algebra itself , but in terms of the *shift of referential* that is necessary to operate within the Numerical field of reference.

(ii) the N-A framework is concerned with the process of solution, not with the problems to be solved or the situations to be structured. As a result, the use of algebraic (literal) notation does **not** characterise any of the two modes. Although solving a "purely" algebraic problem using algebra (eg, formally solving an equation written in symbolic notation) is certainly an activity that develops within a Numerical field of reference, the same "purely" algebraic problem might be solved within an Analogical field of reference (for example, modelling it with a scale balance). Also, the general description of the number of, say, dots on a geometrical pattern "using letters", for example, is typically Analogical, because the choice of operations to be used in the description depends only on the way in which the pattern is visually perceived, but a "purely arithmetical" problem can be handled in a typically Numerical way (eg, $[157+157+157+157+157] \div 5 = 157$ because there are five 157's, etc.).

3.From the historical development of Algebra

Westernly, the historical development of Algebra has been referred to as a succession of three phases: rethorical, syncopated and symbolic (Joseph, 1988, is a brief but excelent appraisal of Eurocentrism in Mathematics). The first phase is associated to pre-greek 'algebra', the second with the work of Diophantus and the third with the work of Viete and Descartes. (eg, Hogben, 1957). This description clearly corresponds to a development of algebra as a subject-matter, given our modern definition of Algebra as a form of "symbolic calculation", and this is thoroughly expressed on the usual assertion that Viete was the first to produce "truly" algebra.

Jacob Klein's work (Klein,1968, originally published between 1934 and 1936) radically departs from this line of analysis. It shows, based on a deep reading of Greek classical texts and on a careful study of Viete's work and of the cultural and conceptual context surrounding him, that Viete's deeper achievement was **not** simply the development of a symbolic notation (his, after all, was to some extent still "syncopated" and full of geometrical suggestions...), but shifting algebra from "solving problems" to "a method for solving problems" . Viete himself comments on his work saying "TO LEAVE NO PROBLEM UNSOLVED". The way in which Viete achieves his goal is by bringing the solution of the problems entirely into the context of numbers and for this reason his work is about how to proceed within a (general) numerical context. Klein's work, however, does not consider similar developments outside the Diophantus-Viete axis.

The work of arabic mathematicians from al-Khwarizmi (c.800) onwards share the same Numerical character of Viete's, and if in many instances careful attention is paid to the process of 'translating' the problems into a suitable Numerical form (Rashed,1984, p20), this does not mean that "solving problems" was the 'raison d'etre' of their work . In fact, the arabic algebra extends itself over "algebraic" powers, operating with polinomials, normal form of an equation, polinomial equations of higher degree, and a number of topics in Number Theory, a body of knowledge that makes Viete's "Introduction to the Analytical Art" look like a first book in school-algebra . **It has to be stressed however, that until at least the 12th century the arabic algebra is totally "rethorical", and even the work of al-Qalasâdî - 15th century - is still in a "syncopated" form** (for example, the use of distinct symbols for x and x^2).(Cajori,1928, items 115,116,118,124)

The nature of the mode of thinking that generates such knowledge is partially explained in the words of an arabic mathematician – As Samaw'al (12th century) – who said that algebra was concerned with "...operating on the unknown using all the instruments of arithmetics, in the same way in which the arithmetician operates on the known [values]" (Rashed, p27). This comment is better understood in the context of the process of "arithmetisation" which algebra underwent after the pioneer work of al–Khwarizmi, a process that consisted in restricting the methods of algebra to those of "arithmetics" (Rashed, p32, but also analysed in many other places in the book. It is particularly interesting to consider the link that Rashed establishes (p25) between al-Khwarizmi restricting himself to equations of the 1st and 2nd degrees and his conception of proof [to a great extent geometrical]). The process of "arithmetisation" undergone by algebra in this period corresponds, in the context of the epoch, to the process of "abstraction" that algebra underwent during the 19th and 20th centuries: the substitution of a collection of procedures for solving "classes" of problems (later: a collection of results about specific systems, "arithmetical" and "non-arithmetical") by a method that allows us to attack problems in any of those classes (later: an "abstract" system the results from which can be applied to all those particular instances of systems). Algebra becomes an autonomous discipline (later: Abstract Algebra becomes an autonomous discipline).

A less explicit – but equally distinctive – aspect of the arabic algebra, is the fact that once a "contextualized" problem is represented in terms of arithmetical relationships, the process of solution develops entirely within the Numerical field of reference. It is for this reason that careful attention is given to the process of "translation": from that point on, the "context" would not provide a source of reference; if the arithmetical relationships do not accurately correspond to the problem, the algebraic method could not detect the mistake and the Numerical process of solution would result in a waste of time (to say the least). This "internalism" is made possible by the development of algebra as a "theoretical" discipline (Rashed, p20) – already clear in al-Khwarizmi's use of normal forms of equations – at the same time it makes possible further developments in algebra. As Klein points out throughout Part II of his book, this kind of "internalism" was not possible in Diophantus, especially because of his conception of number (the conflict between the "pure" number and the "number of things" and the concept of *eidos* as the only possible form of "general number").

Those two principles – "arithmeticity" and "internalism" – are also characteristic of Viete's work, and to such an extent implicitly taken by him that they become almost transparent by staying always in the background of the symbolic invention. However "hidden", these are exactly the principles that support Viete's creation of a "symbolic calculus". (for those who wishfully think that Vieta's algebra is totally context-free, let us remember that he had different symbols for subtractions where one number was known to be greater than the other and subtractions where this was not known)

What becomes evident with this picture in view, is that a content-driven approach to understanding the Algebraic mode of thinking leads us to miss the point that the "symbolic calculus" of algebra was but a consequence of the development of a body of knowledge that already embodied the calculus (*hisab*, for al-Khwarizmi) that is **progressively** made "symbolic".

We think that it is totally adequate, then, to characterise **Algebraic Thinking** as **the mode of thinking that produced – from the arabic mathematicians on, to our knowledge – the "theoretical" discipline we know as Algebra**. As a consequence, "arithmeticity" and "internalism" are features of thinking algebraically. As we said before, "abstraction" would replace "arithmeticity" in a more general characterisation, but we will keep the latter for two reasons:

(i) Our primary interest is in the development of an algebraic mode of thinking; school-algebra is an algebra of numbers, as Algebra was for a very long period of time ;

(ii) We think that by using "abstraction" one reinforces the idea of an absolute "lack-of-meaning", which we deny as misleading.

4. From Algebra as a subject-matter in Mathematics

A simple way of defining Abstract Algebra is to say it is "the study of algebraic systems", an algebraic system being composed by a set, one or more algebraic operations defined on it and a set of axioms which have to be satisfied by the operations. An algebraic operation on a set A however, is a function from A^n onto A, and this means that the set A is mentioned separetely not because its elements are relevant in any sense, but because we want all the operations to refer to the same set. This is, in a sense, the result of the evolution of the "internalism" mentioned in the previous section: the operations are defined internally and they all refer to same set of elements; no other reference is needed. Because we do not want to refer to anything else "external" (particular), the elements are "abstract", and the only way to do any kind of manipulation within this system is on the basis of the properties of the operations. This allows us generality, as operations are "globally" defined. In a very similar way, if one is solving an equation in a "purely numerical way", one has to do it on the basis of properties of the arithmetical operations.

This characteristic of Algebra means that **in Algebra operations become objects, ie, they are a source of reference, they have properties**. This is true both for "number algebra" as it is for Abstract Algebra.

When dealing with school-algebra, it is usually useful to think in terms of operators (eg, "÷2") instead of in terms of bynary operations (Kirshner, 1987), but this does not essentially alter our point, because the operators are built from the arithmetical operations. Moreover, as a consequence of Algebra being used as a method, ie, generally applicable, we are left in fact with only four arithmetical operators (viz., **+a, -a, xa, ÷a**).

This analysis of Algebra as a subject-matter helps us to understand an aspect central to much of the discussion about Algebraic Thinking: that of meaning.

When a problem or situation is modelled in terms of arithmetical relationships, the objects that provide information on "what can be done to manipulate those expressions" are, as we saw, the operations and their properties, this corresponding to an algebraic treatment. On the other hand, when an Analogical model is used the numbers are associated, as "measures" (or operators operating on "measures", eg, "3 buckets"), to some other object; if one is dealing with a "purely numerical" problem, the numbers might be associated, for example, to parts and wholes; those other objects and their "qualitative" structure are the elements which provide us with information on "what can be done to solve the problem". One knows which operation to perform and with which numbers because each operation corresponds to an evaluation and the numbers are "attached" to the parts involved.

What is "lost" in a Numerical process of solution is exactly this Analogical reference on "what to do with the quantities", and this is the meaning of "meaningless" that could be applied to an algebraic solution. ("it is meaningless" $\Leftrightarrow$ "I can't see how those elements tell me this is what I should have done")

6. The N-A framework and research on Algebraic Thinking

(I) Harper (1987) analysed solutions to the problem "If you are given the sum and the difference of any two numbers, show that you can always find out what the numbers are", and

identified three groups of answers that correspond to "rethorical" (totally verbal), "Diophantine" (or "syncopated"; symbols only for the unknowns) and "Vietan" (or "symbolic"; symbols for the unknowns and for the given [general] values) answers.

One has to notice however, that all three kinds of solutions are general, in the sense of being generally applicable to any sum and difference given **and they are thus undistinguishable from that point of view.** Moreover, Viete's answer to the problem (p88) totally corresponds to the "rethorical" answer presented on p81, apart, of course, the use of letters (and this is correct even to the extent that Viete's answer is $^1/_2D - ^1/_2B$ and not $^1/_2$ [D-B]). Whenever a correct "rethorical" answer is not accompanied by an explanation as to how the result was obtained (as it is the case with the one presented on p81), one has to consider that the process of "thinking out the problem" (p80) could correspond to anything, including Viete's method.

The important point here is that although lacking symbolic generality, "rethorical" and "Diophantine" solutions might eventually involve much of the same mode of thinking that a "Vietan" solution does (we emphasised the "eventually" because an Analogical solution is also possible on all three 'styles').

Harper's classification of solutions is certainly useful to describe differences in the use of mathematical symbolism, but by itself it does not provide a framework that enables us to distinguish different **modes of thinking**.

As a consequence we are again led to the necessity of a framework that takes into consideration the ways in which solutions are produced, ie, which are the sources of reference used, and this is exactly the focus of attention of the N-A framework.

(II) Lesley Booth's follow-up study of the CSMS survey (Booth, 1984) produced a number of important findings. Although primarily concerned with situations that involve the use of letters, Booth's conclusions point out to the necessity of understanding children's "informal" methods if we are to understand the nature of the gap between non-algebraic and algebraic modes of thinking.

Of particular interest to us is her characterisation of the "child methods"(p37): "(1) intuitive, ie, based upon instinctive knowledge: not systematically reflected upon and not checked for consistency within a general framework; (2) primitive, ie, tied closely to early experiences in mathematics; (3) context-bound, ie, elicited by the features of the particular problem; (4) indicative of little or no formal symbolized method; (5) worked almost entirely within the system of whole numbers (and halves)".

If those "methods" are seen as based on a qualitative analysis of the situation presented (an Analogical approach), the first four characteristics follow as a consequence: context-bound because the solution depends on understanding a particular situation and the possibility of manipulating its elements to perform evaluations; non-systematiized because of the obvious "one-off" (or even "few-off") character of the solutions; intuitive because non-systematic, but also probably because the knowledge required to perform the qualitative analysis is not seen as mathematical knowledge; little or no symbolization both because the strategies actually used to "think the problem out" – comparing, decomposing and recomposing wholes, for example – are easily and accurately described by verbal statements, and because "thinking the problem out" (using the strategies) is of a different nature than "working the problem out" (the actual evaluations, the performance of the operations). Symbolic notation might be used to **describe** but this does not contribute to the process of solution itself . [This is not the case with a Numerical solution, because the operations are at one time the source of reference and the instruments used to manipulate the information: a concise and homogeneous notation which is intended to be manipulated is adequate and possible] Those "methods" are primitive because the operations can retain their original role, that of being **tools for evaluation**. (the latter idea is also conveyed, in a slightly different form, in the assertion that children see operations as "something to be performed" [Booth,*op. cit.* , pp90-91]).

Three of Booth's research findings (pp85 and following) also provide evidence that an Analogical approach is probably preferential to those students (the item numbers correspond to the original text):

(1.c) " Some children are confused over the distinction between letters as representing the value(s) or number(s) relating to a measure or object, and letters as representing the measure or object itself. ...". From the point of view of the N-A framework, this could be interpreted as a consequence of the students operating Analogically, ie, as the numbers are "numbers of things" and as those "things" are the source of reference on <u>what to do</u> or on <u>how it works</u> (more specifically, the qualitative structure involving those "things"), it would be more natural to represent primarily the "things" and not the numbers that correspond to them.

(3.b.i) "The context of the problem determines the order of operation" and (3.b.ii) "In the absence of a specific context, operations are performed from left to right". Those two points indicate the extent to which the operations are **not** constituted as **objects** and their use remain subjected to other sources of reference.

(III) On a exploratory study carried out in Nottingham, England, in 1989 and reported in Lins(1990), two groups of 3rd year secondary school students and a group of 4th year primary school students were asked to solve a set of five "verbal" problems and to explain <u>why</u> they did it that way. Both correct and incorrect solutions, together with the explanations, were then analysed to determine – whenever possible – the source(s) of reference used by the students to work the problems out.

Two of the problems used:

Carpenter: The stick on the top is 28cm longer than the one in the bottom; altogether they measure 160cm. How long is each of them?

Buckets : From a tank filled with 210 liters of water I took 3 full buckets. Now I have only 156 liters left. How many liters go into a bucket?

The analysis showed that in many cases the solutions were Analogical (eg, "to take 156 away from 210 <u>to determine how much was taken by the 3 buckets</u>" on BUCKETS), but it also showed that in those cases where only the calculations were provided they corresponded – in all but one instance – to those that would be used with the simplest Analogical solution (for example, when solving the Carpenter's problem, to begin with 160–28 but not with 160+28 and never representing the difference as the result of a subtraction [as in **x–y=28**]). The overall result of the analysis suggests that: (i) the use of an Analogical approach, as we define it, is experimentally generally verifiable, and (ii) those students used mainly an Analogical approach.

The following fragments of an interview from another study (Laura , 10yrs5mths) provide a clear example of the use of an Analogical approach: (the problem is "George and Sam have £1.60 altogether, but Sam has 38p more than George does. How much does each of them have?"; the emphasis on the transcription is ours)

Int:... how did you know that you had to take 38p away and not to add 38p?
Laura: If you added 38p... then... ahnn... if you added 38p then you wouldn't have, ahnn... <u>you would have more than £1.60 to start off with... and it says you only have £1.60</u>.
....
Int: But if you take 38 away, then you have less than you had...
Laura: yeah... I think I was just trying <u>to get the 38p out of the way for a bit</u> ! and then...

<u>Features</u> of the situation act as constraints and source of reference in the process of solving the problem.

(IV) In another study, we investigated the sources of reference used by six postgraduate students in the University of Nottingham to <u>validate</u> given symbolic representations as correctly describing a verbally given situation (a brief discussion is in Lins,1988). One of them was the well known "students and professors" situation ("In a school there are six students for each professor,...", etc.). Two basic strategies were identified: (i) always to refer back to the verbalised situation, and (ii) to determine one correct symbolic representation and from it to derive the correctness or incorrectness of the others on the basis of algebraic manipulation. One of the students, who otherwise always referred back to the text and adopted as correct the – "wrong" – representation 6S=P, when faced with the item 18P=3S simply divided both sides by 3 to obtain 6P=S and concluded it was not in agreement with the verbal description. Moreover, she proved quite able to solve formally set equations and had no difficulty with the CSMS item "Which is greater: 2n or n+2". The information gathered by this exploratory study suggests that using an Analogical approach (in that above case modelling the situation by putting "blocks" into

correspondence) is **not** necessarily the result of an inability to deal with "unclosed" or "symbolic" expressions, but rather the result of structuring the situation using a referential that is different from the referential that would produce a representation in terms of arithmetical relationships.

(V) Friedlander et al.(1989) investigated, among other things, differences between visual and numerical justifications, a distinction that corresponds – in the context of the problem analysed, a "geometrical" problem – to our N-A distinction.

<u>7. Conclusion</u>

The N-A framework was developed as part of our effort to provide a clear characterisation of Algebraic Thinking. On its foundation is the assumption of two distinct fields of reference (**Numerical** and **Analogical**).

<u>Numerical</u>	<u>Analogical</u>
Operating within the Numerical field of reference means that only the arithmetical structure is relevant.	Operating within the Analogical field of reference means tha the relevant information is provided by the "qualitative structure (eg, bigger/smaller, decrease/increase, wholes/parts)
the objective of any manipulation is to derive new arithmetical relationships that, because of its form, bring with it new information about the initial relationships; In doing so, one is guided exclusively by the operations involved and their properties. **Operations can have properties because they are OBJECTS.**	the objective of any manipulation is to make **evaluation** possible; this is done through the manipulation of th elements of the situation; comparing wholes an decomposing wholes and rearranging the parts thus obtaine are typical Analogical strategies. **Operations are the TOOLS with which the evaluations are carried out.**
because the guiding principles apply irrespective of the particular arithmetical structure dealt with – with the few canonical exceptions that also apply to arithmetics, like division by zero, etc – **operating within the Numerical field of reference has a strong character of method**; <u>meaning belongs thus to the process as a whole</u>. (A METHOD TO SOLVE PROBLEMS)	operating within the **Analogical field of reference** i an activity bound by the specific "qualitative" structure, an thus presents itself as **a procedure**; <u>meaning belongs to each step of the solution process, as the "qualitative" structure changes with each new evaluation</u>. (TO SOLVE A PROBLEM)
Limits of the context are taken as limits for the answer but not for the process of solution	Limits of the context are taken as limits for the process o solution

Other important general features of the N-A framework are:

(1) <u>The use of symbolic notation is not characteristic of operating within any of the two fields of reference</u>; nevertheless, a symbolic notation that is intended to be manipulated is possible and adequate when operating within a Numerical field of reference but not when operating within an Analogical field of reference.

(2) The <u>central</u> distinction being made is between ways of interpreting the elements of problems and situations and **not** between the consequences of different interpretations;

(3) It avoids the idea of "pre-algebraic" and "algebraic" modes of thinking that is inherent to the content-driven arithmetical-algebraic distinction; this offers us a perspective of analysis of the learning process different from that of developmental stages.

From the point of view of the N-A framework, Algebraic Thinking is naturally defined as the mode of thinking that enables one to operate within the Numerical field of reference.

Nevertheless, Algebraic Thinking applies to fields of reference other than the Numerical (applied to sets it might lead for example to Boolean algebra); for this reason it is adequate to use Numerical instead of Algebraic field of reference, once we are examining the development of Algebraic Thinking in the context of school-algebra, which is certainly an algebra of numbers.

Also, algebra being the study of the properties of an algebraic system (as defined in section 3) Algebraic Thinking is the mode of thinking that leads to the development of algebra, and the symbolic system that corresponds to the calculus embodied in the ideas of algebra is a possible consequence of thinking algebraically, **not** a characteristic of it.

The N-A framework enables us to examine the development of an algebraic mode of thinking in more depth, both because it links Algebraic Thinking to a field of reference (and then – as a consequence – to what is possible and necessary when thinking algebraically) and because non-algebraic thinking is characterised in itself and not as "inability-to-think-algebraically". This positive characterisation of a non-algebraic mode of thinking is essential if we are to understand the "misconceptions", "failures" and "rejections" related to the learning **and** use of algebra. Also, the N–A framework provides a non-circumstancial explanation for the inadequacy of "algebra as a language", by exposing the impossibility of a "translation" producing by itself the required *shift of reference* that takes one into the Numerical field of reference.

Because Numerical and Analogical fields of reference are distinct, operating within one of them cannot be reduced to operating within the other. This means that each of them provide distinct approaches that are more or less adequate depending on the task in hand; non-algebraic approaches are not weaker *a priori* (see, for example, Janvier, 1989 and Fischbein, 1988) and the fact that this conclusion follows from the way in which our definition for Algebraic Thinking is built is certainly an indication of the its adequacy.

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