A FRAMEWORK FOR UNDERSTANDING WHAT ALGEBRAIC THINKING IS

by

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Chapter 4
Experimental Study

"Batatinha quando nasce,
esparrama pelo chão.
Menininha quando dorme,
põe a mão no coração."

Brazilian nursery rhyme

4.1 INTRODUCTION

As we have indicated in Chapter 1, the main objectives of our experimental study are two:

(i) to investigate to what extent our characterisation of *algebraic thinking* enables us to distinguish between different types of solutions for "algebraic verbal problems," and,

(ii) to ascertain the nature of the non-algebraic models used to solve those problems.

The choice of "algebraic verbal problems" as the basic type of problem to be used, is due, first, to our interest in examining the extent to which the *situational context* of a problem may suggest a model or impose unnecessary restrains on the chosen models. Second, *algebraic thinking* involves a shift towards "modelling in numbers," and by using contextualised problems we would be able to discern more shades of the solution process, as the amplitude of the shift would be greater than if we used "pure number" problems. Third, "algebraic verbal problems" are material typically used in the later series of primary school and early series of secondary school, a period of schooling in which we have particular interest; by using our framework to examine that material, we would be, at the same time we conducted the research more closely connected with the thesis's objectives, furthering our understanding of that specific type of problems.

We decided to include "secret number" problems in order to investigate whether the absence of a *situational context* would lead the students to use an algebraic, or at least a purely numerical model, or whether they would try to model the problems by interpreting them "back" into some *situational context* or into some non-numerical *Semantic Field* (eg, whole-part models or geometric models); by using a syncopated notation—abbreviations for the variable names and the conventional symbols for the arithmetical operations and the equality—we would be able to examine how the non-algebraic solvers would make sense of the "arithmetical" context[1], and understand some of the difficulties involved in making sense of a problem presented in that form. This is an issue of particular interest for research on the learning of algebra, and by avoiding the use of "letters" we would be able to focus on the value of the "arithmetical" expressions as *informative articulations*, ie, (local) structures which inform the solution process.

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[1]We use quotes in order to emphasise that we are only referring to a form of *presentation*—as opposed to a form of *representation*. Whether or not the solver will deal *arithmetically*, ie, *in numbers only*, with the problem, is something which cannot be predicted *a priori*.

THE EXPLORATORY STUDY

The object of this small scale investigation was to study the strategies used to solve "algebraic verbal problems" by subjects with little or no instruction in school algebra. Its aim was to understand to what extent the strategies of school algebra are compatible with or similar to those informal solutions, and what kind of obstacles would have to be overcome if one wanted to build a knowledge of school algebra from those informal strategies.

The exploratory study was carried out with three groups. Two third-year groups, 3T and 3A (19 students in each) were from Fernwood Comprehensive School; a younger group, on the last year of primary school, J (21 students), was from Fernwood Junior School. Both schools are in Nottingham, England.

Group 3T was rated as top-ability by the school; group 3A was rated as low- to average-ability.

The test presented to J and 3T consisted of five "algebraic verbal problems," plus two questions about "making change". The test presented to 3A consisted of different versions of four of those five problems, plus the remaining problem with the same text, plus five short questions about solving problems.

Each problem corresponded to a different "algebraic structure," ie, it would correspond to a different type of equation.

Both sets of problems are presented in Annex A.

Of the five main problems used in this study, only one, the "Consecutive Numbers" problem, was not used in the main study, primarily because its investigative nature required more time for it to be solved. The specific results of the exploratory are in complete agreement with those obtained in the main study—which are presented in the subsequent sections—and for this reason will not be discussed here.

The only remark which is worth making is related to the "Consecutive Numbers" problem, which was not, as we said, used in the main study. Unexpectedly, the primary school students performed equally well as, if not slightly better than, the secondary school students. Given the very small size of the samples, this information cannot be taken as indicative of any general phenomenon, but we were led to believe that the students in J dealt more freely with the problem, ie, apparently they had less expectations about how this type of problem "should" be solved, both because the problem was completely new for them, but also because their experience with solving problems was much less related to the use of specific methods, and as a consequence they were more able to explore the situation

The six test papers composed, in fact, three pairs of test papers; each pupil was presented with one of the pairs, each test paper presented in a session, never on the same day, and never more than a week later. Each paper was solved in a 50 minutes session.

An important aspect of the testing conditions, was that the students were allowed to use calculators whenever they were available, as well as being told, **in all cases**, that the calculations could be just indicated if the student thought it was "too hard" to perform. They were told, moreover, that they could solve the problems using whichever method they wished, and the word "algebra" was carefully avoided in the introductions, in order to prevent induction to a specific method, but also to prevent causing anxiety in those students who knew little or nothing of "algebra."

The particular aspects of each group of problems examined in this dissertation are presented in the relevant sections on the data analysis.

For the main study we contacted two schools in Brazil—*Escola de Aplicação da USP* and *Colégio Hugo Sarmento*, both in the city of São Paulo—and two schools in England—*Friesland Comprehensive School* and *Margaret Glen-Bott Secondary School*—both in Nottingham. We decided to work both with Brazilian and English groups for two reasons. First because the marked differences in the teaching of mathematics in the two countries—in method as well as in content[3]—suggested that we would have a much more varied sample in terms of approaches and models used, a suggestion which proved to be correct. Second, because we would have the opportunity to carry out a preliminary investigation into the effect of different teaching approaches in the development of an algebraic mode of thinking, an aspect which we intend to further examine in the future.

Two Brazilian 7th grade groups (age 13-14 years, 56 students), two Brazilian 8th grade groups (age 14-15 years, 53 students), three English 2nd year groups (age 13-14 years, 53 students) and three English 3rd year groups (age 14-15 years, 66 students), form the sample of the main study. The number of students and the average age for each group, are given in Annex C.

As a consequence of the test papers structure, each question was solved by roughly one-third of all students in the sample (total of 228 students).

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[3]The teaching of mathematics—particularly the teaching of algebra—in Brazilian schools is, almost invariably, content-driven and quite formal; investigative activities are very rare in Brazilian mathematics classrooms. One may safely say that quite the opposite is true in English schools. This general picture applies very well in the case of the four schools where our experimental research was conducted.

Five categories were used to classify the solutions:

1) correct solutions in which the problem is solved by setting and solving a numerical equation in a recognisable form (**OKEQT**);
2) correct solutions that did not use any recognisable form of equation; the calculations used to produce the answer are presented, with or without an explanation or a diagram supporting the choices of calculations to be performed (**OKCALC**)
3) incorrect solutions where there was an attempt at using an equation (**WEQT**);
4) incorrect solutions where equations are not used; calculations are presented, with or without an explanation or a diagram supporting the choices of calculations to be performed (**WCALC**);
5) trial-and-error solutions (**T&E**);

Calculations wrongly performed did *not* characterise a solution as "incorrect": if the overall procedure would lead to a correct answer had the calculations been performed correctly, the solution was classified as "correct"; also, there were cases in which a complete answer involved the determination of two values and only one of them was given by the student: the correctness of the solution in those cases was assessed in relation to the potential of the method employed to produce the second value, and in relation to the student's awareness of the existence of two values to be determined, as shown in the establishment and manipulation of the chosen model.

The categories above are intended to describe only the form of ***presentation*** *of the solutions, not the underlying model*; an **OKEQT** solution, for example, does *not* imply the presence of *algebraic thinking*. We consider this set of categories to be suitable for two reasons: (i) on the one hand, it is *standard*, providing categories which are easily understood and applied by other people; and, (ii) precisely because it is based on the perceived proximity of a solution to "standard algebraic solutions"—notationwise—the analysis of scripts belonging to a same category allows us to highlight the importance of understanding the underlying model in the process of investigating the nature of the thinking involved in producing a given solution.

In this sense, the categories above provide a general "background" framework, which is not supposed to correspond to the much finer understanding which is produced by the analysis of the scripts. Moreover, in the examination of the scripts, we have not characterised them according to the polarities produced in Chapter 3, from the historical study. The analysis conducted in Chapter 3 has a much more dynamic nature than that conducted in the context of the experimental study, mainly because in Chapter 3 we not only elicit the models accepted by a given mathematical culture, but also relate the acceptance of those models to the more general conceptual framework of the mathematical culture in question; in the case of the experimental study, the application of a similar type of analysis would necessarily involve examining the mathematical *ethos* of those students—a line of research which seems to belong naturally to future extensions of our present work. Attempting to use the polarities from Chapter 3 to produce some sort of justification of the choice of models we had identified, seemed, thus, an artificial and inadequate approach.

Although recognising the importance of providing a more complete and "actual" framework for characterising the non-algebraic solutions, we think that it would not be possible to produce such a framework in the context of this dissertation, above all because it would depend on a much deeper study of modes of thinking other than the algebraic one.

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For the purpose of our analysis, four groups were considered: **AH7**, which comprises all the Brazilian 7th grade groups; **AH8**, which comprises all the Brazilian 8th grade groups; **FM2**, the English 2nd year groups; and **FM3**, the English 3rd year groups.

All the percentage results of each problem examined in the analysis of the experimental study, given for each of the four groups above, is in Annex D; nevertheless, those percentages which suggest relevant or interesting aspects of the overall solving activity, are quoted again in the the section corresponding to the group of problems to which they refer.

The methodological approach of our analysis of the data gathered in the main study is thoroughly *qualitative*; this means that no strong claim is made exclusively on the basis of the percentage results, but also that no statistical treatment was applied to the percentage data. In our analysis, the percentage data only *suggests* underlying modelling trends, and any claim is supported by instances to be found in the scripts.

4.2 Ticket and Driving Problems

The Problems

Sam and George bought tickets to a concert.
Because Sam wanted a better seat, his ticket cost four times as much as George's ticket.
Altogether they spent 74 pounds on the tickets.

What was the cost of each ticket?
(Explain how you solved the problem and why you dit it that way)

Tickets 4x

Sam and George bought tickets to a concert.
Because Sam wanted a better seat, his ticket cost 2.7 times as much as George's ticket.
Altogether they spent 74 pounds on the tickets.

What was the cost of each ticket?
(Explain how you solved the problem and why you dit it that way)

Tickets 2.7

Mr Sweetmann and his family have to drive 261 miles to get from London to Leeds.
At a certain point they decided to stop for lunch.
After lunch they still had to drive four times as much as they had already driven.

How much did they drive before lunch? And after lunch?
(Explain how you solved the problem and how you knew what to do)

Driving 4x

Mr Sweetmann and his family have to drive 261 miles to get from London to Leeds.
At a certain point they decided to stop for lunch.
After lunch they still had to drive 2.7 times as much as they had already driven.

How much did they drive before lunch? And after lunch?
(Explain how you solved the problem and why you did it that way)

Driving 2.7

GENERAL DESCRIPTION

This is the only pair of problems to appear on all three sets of questions, with the pair Tickets [4 times] (**T4**) / Driving [2.7 times] (**D2.7**) appearing in the Blue-Gray and Green-Beige tests, and the pair Tickets [2.7 times] (**T2.7**) / Driving [4 times] (**D4**) appearing on the Yellow tests.

The questions were designed to investigate to what extent different kinds of numbers — namely, counting numbers vs. decimal non-integer numbers — would affect the choice of models used to solve problems with the same "algebraic" structure, and which models would result. The [4] problems have the structure "this is 4 times as much as that, and altogether...", and the [2.7] problems have the same structure with 2.7 replacing 4.

In order to have some control over possible effects of the context in which the problems were set, we used two contexts with different characteristics. In the "Driving" problems the objects are portions of a road with different lengths, which can be sectioned (for example, to be compared) and still maintain their characteristic as a portion of a road. In the "Tickets" problems the objects are tickets with different values; there is no real meaning in "sectioning" one of the tickets, and any direct contextualised comparison would have to be made on the basis of the exchange values. It is clear that in both cases a comparison is possible using respectively the lengths and the values.

DISCUSSION OF POSSIBLE SOLUTIONS

The simplest algebraic model that fits into those problems is a linear equation in one unknown . A direct "translation" from the problems would in fact produce a set of two linear equations in two unknowns. In *Tickets and Driving* , however, this representation was never used; instead, *direct substitutions* were used, which we will comment a few paragraphs ahead.

Depending on whether the unknown (here represented by x) is taken as the cheaper ticket or the distance travelled before lunch, or as the more expensive ticket or the distance travelled after lunch, we would have one of the following equations:

(E1) $\mathbf{x + ax = b}$

(E2) $\mathbf{x + x/a = b}$

(E3) $\mathbf{b - x = ax}$

(E4) $\mathbf{b - x/4 = x}$

with the corresponding values of **a** and **b**.

Equations E2 and E4 were never used by any student. Equation E3 was used by one student only.

Setting the equation can be done in two very distinct ways, either by directly representing a **numerical** relationship ("a number plus **a** times this number is equal to **b**") or by representing instead a **whole-part** relationship. On the former situation, the model applies equally both to [4] and to [2.7] problems, because only a knowledge of operating with decimal numbers is required (to multiply, to add — very much as it has to be done with the [4] problems where only counting numbers are involved) and for the students in our study this knowledge was sufficiently developed. On the latter situation, however, producing meaning for "**4x**" and for "**2.7x**" are processes that involve different degrees of difficulty, even if calculating aspects of decimal numbers are well understood.

A **whole-part** model is quite simply produced for [4] problems: "1 (lot of) **x** plus 4 (lots of) **x** is equal to ..."; the 1 and the 4 play their natural role of "counting numbers". When the same model is applied to [2.7] problems, the need to interpret 2.7 as a "counting number" becomes an obstacle because it requires — at least — the additional step of decomposing the "2.7 lots" into "2 lots and 7 tenths of a lot" for the "counting" to become visible.

Alternatively, an analogy could be drawn with "2.7 pounds of beans" (and one would reasonably expect the students in our study to have no difficulty in concluding that "if one buys 1 pound of black beans and 2.7 pounds of chilli beans, one has 3.7 pounds of beans altogether", indicating a willingness to accept decimals as quantifier). However, to successfully apply this analogy to [2.7] problems one has to take the smaller of the two quantities (cheaper ticket or shorter portion of journey) as a *unit*[5].

No matter which model is used to set the equation, an Algebraic solution of the equation is one that is based on properties of the arithmetical operations and of the equality involved in the equation.

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[5] A step not easily seen by those students, as the analysis of the data will show.

A property like $\mathbf{ab = c \Rightarrow b = \frac{c}{a}}$ can be easily justified in terms of "sharing" if **a** is a positive integer ("if **a** lots of **b** is equal to **c**, then sharing **c** into **a** parts will give the value of **b**"), but not otherwise. If however this property is seen as a property of the **numerical** relationship, and thus also applicable when **a** is not a positive integer, we will consider that an algebraic understanding exists, and if the "explanation" is maintained it will be seen as a particular illustration of the property.

A straightforward solution to E1 would be,

(D2.7)

$$\mathbf{x + 2.7x = 261}$$

$$\mathbf{3.7x = 261}$$

$$\mathbf{x = \frac{261}{3.7} \approx 70.5 \text{ miles, etc..}}$$

It is important to observe that the operations performed with **D2.7** would be:

(i) $1 + 2.7$; (ii) $261 \div 3.7$; (iii) 70.5×2.7 ;

and with **D4**,

(i) $1+4$; (ii) $261 \div 5$; (iii) 52.2×4 .

Non-algebraic models that fit into those problems' context would almost certainly be of the type "1 lot and **a** lots, giving...", be they supported by or derived from a line diagram, a Venn diagram, or a block diagram, ie, a **whole-part** model (Figure T&D 1). As we saw above, the structure produced by such models can be reinterpreted as a numerical relationship and manipulated algebraically, to produce an algebraic solution. But such structures can also be directly manipulated, with calculations performed only to achieve required evaluations of parts.

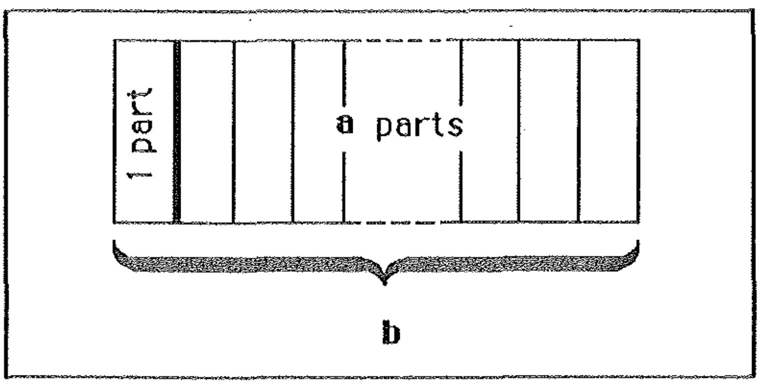

fig T&D 1

With **T4** the manipulation of the whole-part structure would proceed like this:

(i) one of the tickets is 4 times more expensive then the other one; this is the same as saying it "is" 4 tickets;

(ii) 1 ticket and 4 tickets cost **b** pounds, ie, 5 tickets cost **b** pounds;

(iii) now, to know how much 1 ticket costs, I share the **b** pounds into 5 tickets.

With **D4** we would have the same general procedure, with "parts" or "sections" replacing "tickets". It is clear that "lots" would work well with both .

Operations are used to evaluate parts as necessary. Thus,

(ii') 1+4 corresponds to evaluating the total number of tickets, and,

(iii') **b** ÷ 5 corresponds to evaluating how much goes to each of the 5 tickets **through the sharing.**

When the same model is applied to [2.7] problems, two difficulties arise. One is the reinterpretation of "2.7 times more" as "2.7 tickets" or as "2.7 sections". Although the problem is concerned with the value of the tickets, the non-algebraic models deal with this by associating "the **value** of one ticket" to "one ticket", the image of the ticket working as an icon for the value. It is from this point-of-view that the 2.7 should have to "count" tickets in the way the 4 naturally does, with the consequences pointed out a few paragraphs above.

The second difficulty is in fact twofold. On the one hand, there is a problem with step (iii) above. In our description of the *non-algebraic* solution for **T4** we used the word "share" — underlined for emphasis — because we wanted to stress that the main aspect of the manipulation is the sharing, the result of which is eventually made actual either by performing the division by 5, a build-up calculation or by a trial-and-error process. In the case of [2.7] problems, obtaining the value of "1 lot" by "sharing" the total into "3.7 lots (?)" is certainly a difficult and "unnatural" step.[6]

On the other hand, it is difficult to see why anyone would want to step into (ii) without being aware that this is an intermediate step leading to (iii); step (ii) corresponds to "finding how many altogether so I can share between them" instead of "collecting the various occurrences of the unknown". Although in procedural terms step (ii) is processed

[6]"unnatural" to the extent that experts would use such metaphor only to try and make a verbal link with some situation where only "true" counting numbers appear.

before step (iii), both steps are engendered in conjunction: the two aspects are composed to produce a larger obstacle that has to be overcome in one go[7].

One important point in relation to this group of questions is that it is clear here that the use of algebraic symbolism (standard or not) is not enough to guarantee that algebraic processes are involved in the solution of [4] problems. Algebraic notation could be used as a concise notation for a non-algebraic solution, a complete correspondence existing with the steps of an algebraic solution (figure T&D 2), as much as a "calculations only" solution could have been guided algebraically (the problem being simple enough to allow that).

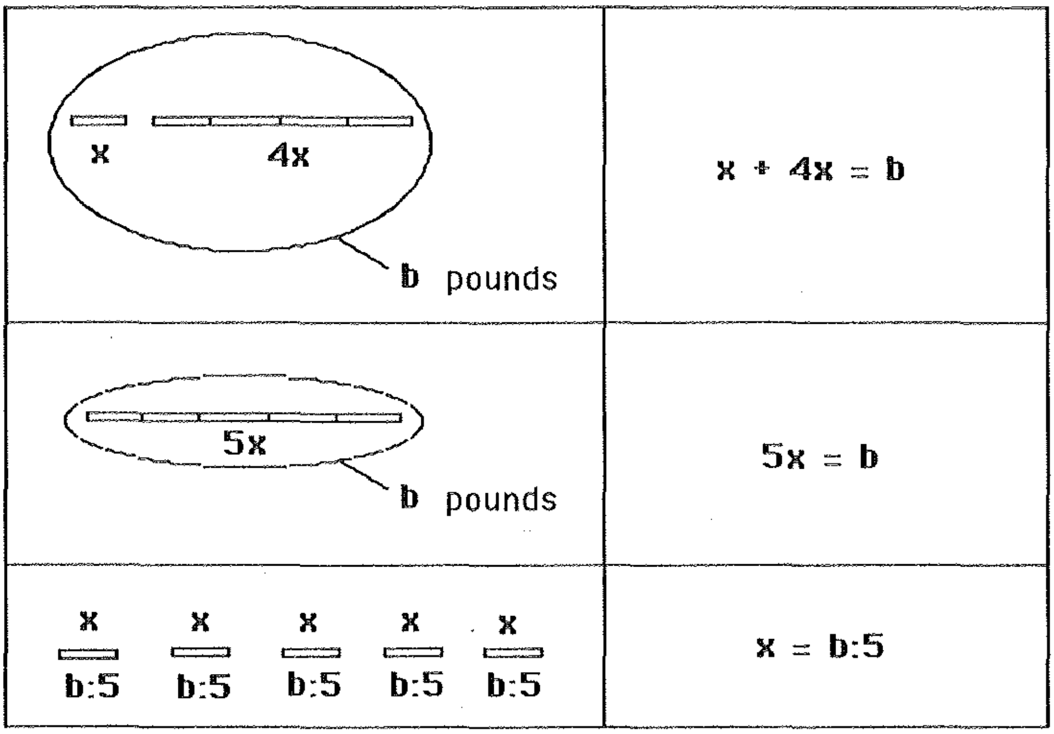

fig T&D 2

Nevertheless, our analysis also indicates that no matter the notation employed, the greater the use of an algebraic model by a group of students would produce a smaller difference between the facility levels for [4] and [2.7] problems.

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[7]The analogy with "buying x and y pounds of ..." would not be enough to overcome alone this double difficulty: the "anticipation" problem would remain.

Previous research on the solution of multiplicative problems has pointed out that the operations of arithmetic (multiplication and division being of interest for us in this section) might remain linked to "primitive behavioural models that influence tacitly the choice of operations [to be used to solve problems] even after the learner has had a solid formal-algorithmic training" (Fischbein et al., 1985, p.3). According to Fischbein, the preferred model for multiplication would be one of *repeated addition*, and the preferred models for division would be those of *partitive or sharing division* and of *quotative or measurement division*. It is clear that "under such an interpretation ...a multiplication in which the operator is 0.22 or 5/3 has no intuitive meaning." (op. cit., p.4)

Our identification of the difficulties that might arise from applying a whole-part model to [2.7] problems is in resonance with the interpretation provided by Fischbein and his colleagues to the difficulties they identified. Moreover, it is an integral part of their interpretation that the "...Identification of the operation needed to solve a problem with two items of numerical data takes place not directly but as mediated by the model" (ibid.), which means that the phenomenon they identified can be examined as an instance of non-algebraic thinking. From this viewpoint, the fact that "...the enactive prototype of an arithmetical operation may remain rigidly attached to the concept long after the concept has acquired a formal status" (ibid., pp. 5-6) is reinterpreted in two ways[8]:

- that the enactive prototype remains attached to the concept (at least in relation to contextualised problems) is seen as <u>a consequence of rather than a cause to the preferential use of non-algebraic models</u>; the properties of the operations that will be reinforced — and will thus remain characteristic of the use of the operations in such situations — are those that correspond well to, for example, whole-part models: Fischbein's repeated sum corresponding to our "counting multiplication", and division as "sharing";
- if what is meant by "acquiring a formal status" is understanding the reversibility of operations, then it is clear that the use of non-algebraic models would account for the observed effect, once something that would be meaningful in the *Semantical Field* of numbers and arithmetical operations has to be blatantly overlooked for the [2.7] problems to have a higher degree of difficulty; if on the other hand it simply corresponds to "...the learner has had solid formal-algorithmic training" as quoted before, it then means that the

[8]The primary aim of reinterpreting Fischbein's findings in terms of our framework is not to add directly to them—although we think we do, but part of our effort to bring together several research findings of interest for the research on Algebraic Thinking, providing a common explanation in terms of our framework.

operations are not used in the problems with this same generality because the models used do not have the required generality, and we have shown that this is the case with whole-part models.

Bell et al. (1989a, p. 438) criticized Fischbein's *Theory of Intuitive Models*, saying that

> "...First, although its basis is the children's assumed perceptions of the structural properties **of the operations**, it can only be made consistent with experimental results by adding an extraneous hypothesis; second, numerical perceptions involving the ignoring of decimal points cause conflict with its predictions. These considerations suggest that the theory gives insufficient weight to pupils' numerical, rather than structural, perceptions" (our emphasis)

and developed a *Theory of Competing Claims* that takes Numerical Preferences as the most significant factor in determining the choice of operation. By considering four possible aspects of solving the problems, rather then focusing in only one as the *Theory of Intuitive Models* does, the *Theory of Competing Claims* produces a much finer analysis, with a much more precise adjustment to the experimental data. It is true, however, that the difference between the results of the two analysis is one of degree of precision rather then one of major conflict[9]. Moreover, the Numerical Preferences hypothesized in Bell et al. (1989a, p. 438) — "...preferences for dividing the larger by the smaller number and for multiplying or dividing by an integer..." — can be put, at least partially, into correspondence with Fischbein's preferred models[10].

There is an important point to be examined here. Both Fischbein's and Bell's models consider only the case where *the operations* have a "structure" (Bell) or "model" (Fischbein) associated to them. But if we are examining the <u>choice</u> of operation, then one of the following cases must apply: (i) the subject solving the problem simply "scans" the list of all calculations — arrangements of numerical data and arithmetical operations — until one is found that seems to be a correct choice, or (ii) the subject produces a model of the

[9] Where Bell's analysis produced four clearly distinct levels of difficulty, Fischbein's analysis produced only two, without however any major inversion on predicted levels of facility, ie, if question **A** is at a lower level than question **B** according to Fischbein, it is never the case that **B** is at a higher level than **A** according to Bell's analysis. (see Bell et al., 1989a, pp. 441-442)

[10] Preference for multiplication by an integer corresponding to the repeated addition model, and preference for division by an integer corresponding to the sharing model.

situation given — in many cases a partial model only — and **on the basis of the model** decides which operations could and should be used; it is only then that this or that operation will be seen as suitable or not. On the first case, numerical aspects — which account directly for three of the four aspects examined by Bell — would certainly constitute a strong factor.

In the second case, we argue that there are two layers of behaviour. At the first level, the subject tries to make sense of the situation and to produce a model that seems adequate. If she or he considers to have found a suitable model, the solution proceeds by manipulation of the chosen model; <u>the use of an operation is suitable or not only in relation to this model</u>, ie, it depends on whether or not using it makes sense in the context of the semantic framework of the model. The solution process might be eventually blocked if the model can not be purposefully manipulated by the subject any further. At a second level, if and when the subject does not produce a model that works in a satisfactory way for her or him, then other aspects come into <u>direct</u> consideration to guide the choice of operation (for example the fact that buying 0.75 pounds of flour must cost less than buying one pound together with the belief that "division makes smaller", makes division a natural choice). This is not to say that such factors play no role in the elaboration of the model, but only that their influence is direct or indirect — and thus more or less diluted — depending on the level one is working at.

This formulation of the process shifts the focus of the analysis from limitations <u>intrinsic</u> to the operations to limitations to their use created by the <u>purpose</u> with which they are used. With non-algebraic models, the purpose would be to <u>evaluate</u> parts as required by the manipulation of the model; with algebraic models, the purpose would be to produce new numerical relationships of required <u>forms</u>, by transforming previously produced relationships; when a structure fails to be produced, operations are chosen as to produce (psychological) <u>contentment</u> in relation to the expected outcome of the problem. It is clear that the last of the three situations is the one where Numerical Preferences — in Bell's sense — are bound to predominate.

Moreover, this approach enables us to understand beyond "arithmetical ability" (performing the operations with different kinds of numbers) the difficulties here examined.[11]

[11]This is a very adequate outcome of our approach. Fischbein (1985, p.4) reminds us that "To say that multiplication by 0.22 or 5/3 has no intuitive meaning is not to say that it has no mathematical meaning. Children may know very well that 1.20 x 0.22 and 9 x 5/3 are legitimate mathematical expressions", and Bell's study (1989a, in particular figure 1, p. 440) shows that although performance improves with age (which most certainly means, in the

The results of a second study presented on the same paper (op. cit., pp. 444-447) also offer some support to our interpretation[12],

> "The making of a correct estimate depends on a correct perception of the operational structure of the problem. This does not necessarily require identification of the numerical operation needed to calculate the exact result. We know from the numerical misconception MMBDS that pupils must have an awareness of the size of the expected answer before making a choice of operation. We suggest that in division problems and problems involving multiplication by numbers less than 1, the estimate is made directly by a semiqualitative ratio comparison, without explicit identification of the division operation".

suggesting that modelling happens prior to the choice of operations.

On the basis of our analysis a local hierarchy can be established for the Tickets and Driving problems:

- if the model used is totally algebraic, with respect to both setting and solving the equation, then the degree of difficulty is the same for all four problems;
- if the model used consists of setting the equation as a description of a non-algebraic structuring, and then solving it algebraically, then [4] problems are easier than [2.7] problems;
- if the model used is purely non-algebraic, then [4] problems are significantly easier than [2.7] problems.

It is against this local hierarchy that we will examined the preferred models used by the students.

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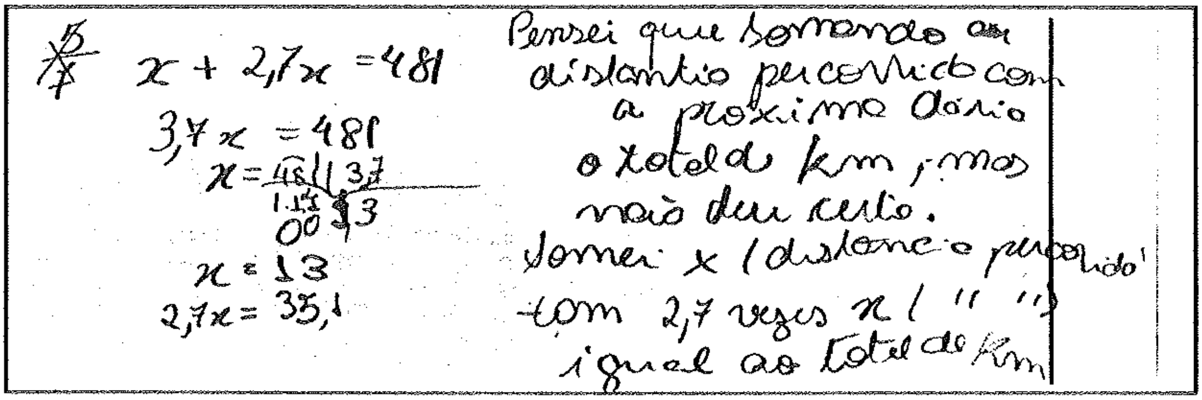
case of the study's sample — all engaged in formal education — improved "arithmetical ability"), similar difficulties occur throughout the whole range of age groups.

[12]This becomes even more clear if one substitutes "... a correct perception of **the** operational structure of the problem" by " ... the perception of **an adequate** operational structure for the problem."

As it is clear from the data, the [4] problems were much more accessible to the students than the [2.7] problems. This is true not only for the overall numbers, but also for each of the four groups.

A possible explanation for such a difference in the facility levels would be that the decimal numbers introduced difficulties with the actual calculations. This is not the case, however, because: (i) errors in the calculations were not considered as errors when the overall procedure would lead to a correct answer were the calculations correctly performed (Alessandra A, A81), and (ii) the students either used calculators or were told that calculations could be just indicated if they felt it was too "hard" to do. There is also the fact that 32% of all wrong answers to **T2.7** and 45% of all wrong answers to **D2.7** resulted from dividing the total by 2.7 instead of 3.7 .


$x + 2,7x = 481$

$3,7x = 481$

$x = 481 \mid 3,7$

$x = 13$

$2,7x = 35,1$

Pensei que somando as distancias percorrido com a proxima daria o total de km, mas não deu certo.
Somei x (distancia percorrido) com 2,7 vezes x (" ") igual ao total de km

Alessandra A - **D2.7**

It is true that the decimal numbers could have affected the use of a trial-and-error strategy. However, the percentages of T&E solutions are very low both for [4] and [2.7] questions, which indicates that this negative effect is totally negligible (in fact, the higher percentage of T&E solutions appears exactly for **T2.7** – 8% overall).

In all four groups, solutions for the [4] problems depended **less** on an algebraic model being used for a correct answer to be achieved, as it is indicated by the fact that the percentages of correct algebraic solutions in relation to the total of correct answers is smaller for the [4] problems than for the [2.7] problems (41% for **T4**, 21% for **D4**, 71% for **D2.7** and 53% for **T2.7**). In **FM2** this is not strictly true because the percentage of correct algebraic solutions for **T2.7** is zero, but given that the level of correct answers is so low (6%) — and all of them obtained through T&E — the dependence on an algebraic model — or to put it another way, the inefficiency of other models — is also established. The same observation is valid for **FM3** in relation to **D2.7**, but not in relation to **T2.7**.

The distinctive aspect in **FM3-T2.7** is that the percentage of T&E correct solutions is much higher than in the other three groups, accounting for 56% of the correct answers. The same group produced no T&E solutions for **D2.7** and one explanation is that the numbers in **T2.7** are far more "triable" than those in **D2.7** . However — and from the viewpoint of our research this is more relevant — the percentage of "÷3.7" (correct) solutions is only 16%, with no correct algebraic solutions, which would produce, were it not for the T&E answers, a very low level of correct answers.

Central in respect to this group of problems, the percentages of correct answers are significantly higher for [4] problems than for the corresponding [2.7] problems, which indicates, in the light of our previous analysis, a clear tendency towards *non-algebraic* models.

This finding is supported in a more direct way by the fact that:

- differences in percentages of "÷3.7 or 5" (correct) solutions for corresponding [4] and [2.7] problems are also very significant (below 25% only for **AH8-T4** and **T2.7**; to **AH8**, however, corresponds the highest percentage of correct algebraic solutions for **T4**, 73%), and
- whenever there is a significant difference in the percentages of correct algebraic solutions to corresponding [4] and [2.7] problems, the balance leans towards the [4] side.

STUDENTS' SOLUTIONS

A number of solutions involved the whole-part models examined in the previous sub-section. With *Tickets* problems this meant for example, stating that "there are the equivalent of 5 tickets in the sum" (David W, F3A; Sergio R, HS8I),

Sam's ticket was £14.80, and George's was £59.20. As there are the equivalent to 5 tickets in the sum, I divided 74 by 5 to get Sam's figure. To get George's, I multiplied that by 4.

David W - **T4**

222 = 5 entradas

222 | 5
44.4

Cada entrada cuesta 44.4

Sergio R - **T4**

and with *Driving* problems, "splitting" the journey into 5 *sections* or *parts* (Elizabeth W, F3B; Clare B, F3B; Jack D, F3B; Jacob B, F3A).

261 ÷ 5 = 52.2 before lunch
261 - 52.2 = 208.8 after lunch

there is 4 more sectons to do after lunch you have done 1 there are 5 all together. 8 261 ÷ 5 = one secton take the one secton from 261 = 208.8 give you the rest

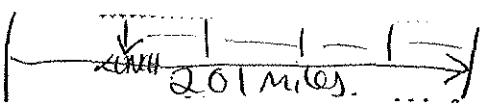
Clare B - **T4**

261 miles
261 ÷ 5 = 52.2 miles before lunch
52.2 × 4 = 208.8 miles after lunch
Then if I add the miles together 208.8
+52.2
261 miles

Jack D - **T4**

I took 261 (miles) and divided by 5 giving 52.2, I then took 52.2 off 261, giving 208.8, now 4 × 52.2 = 208.8, 208.8 + 52.2 = 261, so they drove 52.2 miles before lunch, and 208.8 miles after lunch. I knew this because it is obvious that the journey is 'split' into 5 'parts' (1 before lunch, 4 after) so all I do is divide 261 by 5, and take the result of 261 ÷ 5 (52.2) off 261, giving the last 4 parts of the journey ie 52.2 × 4 = 208.8, 208.8 + 52.2 = 261

Jacob B - **T4**



First I decided it was 4 times as much after lunch, and 1 time before hand, that would equal 5 sections So I divided 261 by 5 which come to 52.2 miles..

So they drove 52.2 miles before lunch and 208.8 miles after lunch.

Elizabeth W - **T4**

The use of diagrams not only shows how parts and sections themselves are taken as objects, but also emphasize how difficult it would be to use this model in a [2.7] problem.

One "calculations only" solution to **T2.7** shows, on the other hand, how close it may be to an algebraic solution that does not employ algebraic symbolism (Nick P, F3B).

2.7 + 1 = 3.7
74 ÷ 3.7 = 20.
20 × 2.7 = 54
54 + 20 = 74.
Sam's ticket cost £54
george's ticket cost £20.

Nick P - **T2.7**

This is a particularly interesting instance: Nick's solutions to a "secret number" problem corresponding to **$6x + 165 = 63$** shows his awareness of treating numerical relationships in purely numerical terms, but nevertheless, his scripts also show that he never spontaneously produced numerical relationships to model problems that had not one already given in some explicit form (the "secret number" problems, for example). Another script, however, shows us the opposite case: Jenny G (F3B) writes down an arithmetical sentence that correctly models the problem, but fails to go any further (supposedly for not knowing how to derive the value of the question mark from that expression).

? + (? × 2.7) = 261

Jenny G - **D2.7**

Each of those students' cases illustrate an aspect of embryonic algebraic thinking: Jenny's awareness of the numerical model; Nick's awareness of the purely numerical treatment of numerical relationships. It is the fusion of those two aspects that produces the algebraic solution in Vanessa J's (F3A) script.

x + (x × 4) = 74.
5x = 74
1x = 14.8
EACH TICKET COSTS £ 14·80

Vanessa J - **T4**

Flavia C (A7I) and Alex K. (A8I) correctly set and solved equations, as did Carolina R (HS8I). It is important to notice, however, that Carolina's equation derives from an initial representation of the problem that is different from Flavia and Ernesto's. While they thought in terms of "what composes the total", she thought in terms of "what is left after the first part of the journey". However derived from different initial readings of a whole-part scheme, the three solutions converge as they reach a point from where they are only concerned with operating within the realm of numbers.

x → Entrada de Samuel
4x → Entrada de Jorge (4 vezes mais que a de Samuel).

$4x + x = 220{,}00$ → equação; x é o preço de 1 entrada.
$5x = 220$ — 4x é 4 vezes o preço da entrada.
$x = \frac{220}{5}$ — 5x é o que todos gastaram. 220 é o que gastaram
$x = 44$ — x = 44, uma entrada custa Cr$ 44,00.

R: A de Samuel custou Cr$ 44,00. A de Jorge, Cr$ 176,00

Flávia C - **T4**

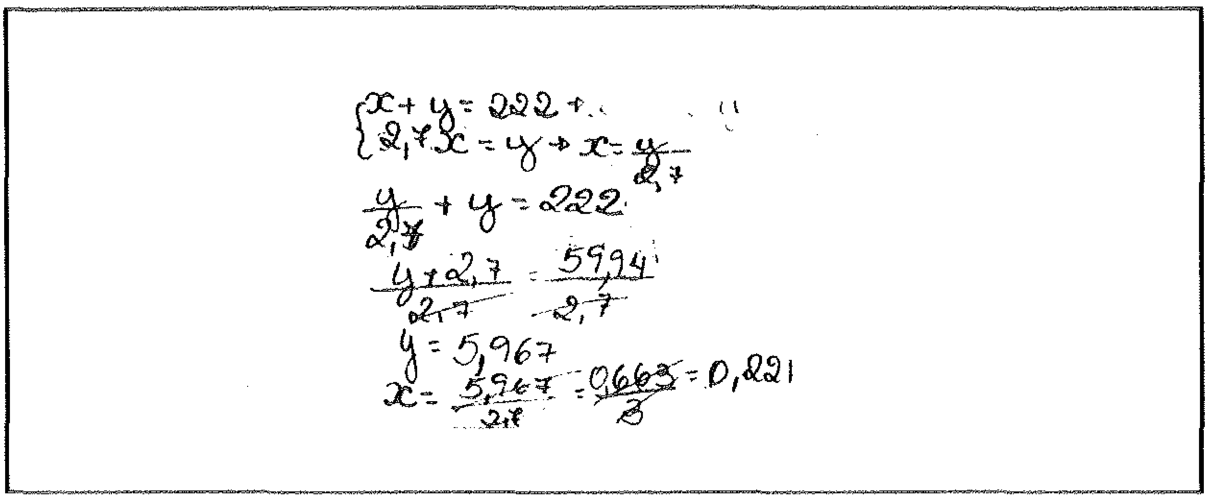
x = antes do almoço (substituindo o nº de quilômetros
2,7 x = depois do almoço (pela incógnita x)
$x + 2{,}7x = 481$ (somando "antes" e "depois" ~~para~~ resulta no nº total de quilômetros)
$3{,}7x = 481$
$x = 481 : 3{,}7$ (passando a operação para o outro lado da igualdade para se isolar a incógnita)
$x = 130$ (total de Km rodados antes
$2{,}7x = 2{,}7 \cdot 130 = 351$ (Total dos Km rodados depois)

Alex K. - **D2.7**

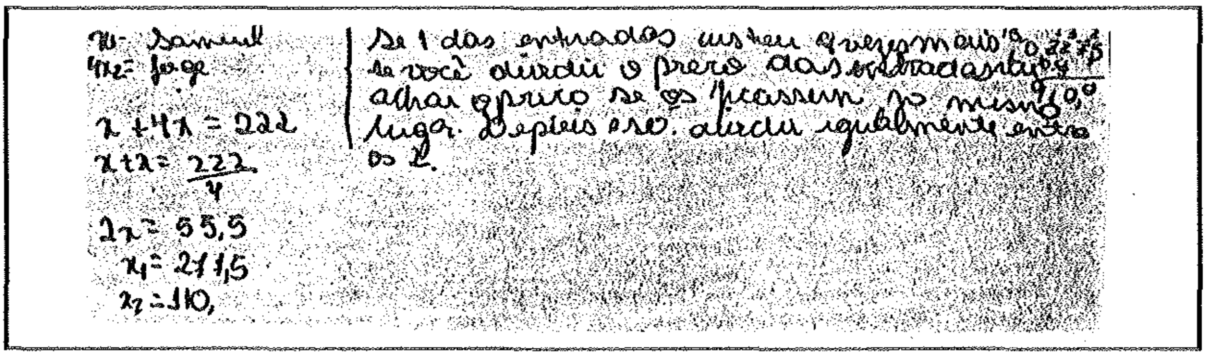
$481 - x = 2{,}7x$ — antes do almoço = 130
$3{,}7x = 481$ — depois = 5.06,9.
$x = \frac{481}{3{,}7}$
$x = 130$

Carolina R - **D2.7**

Another group worth examining is that of wrong solutions in which standard algebraic notation is employed. In two of our examples (Adriana V, A8I; Ana C, A8I), the initial equations correctly model the problem's situation, but they are dealt with in an incorrect way: there are *technical* errors.



Adriana V - **T2.7**



Ana C - **T4**

On the other two examples (Vinícius G, A8I; Adriano I, A8I), the initial equations do not model the problem correctly, but this time they are correctly solved: there are *modelling* errors.

$$x(4+x) = 222$$
$$4x + x^2 = 222$$
$$x^2 + 4x - 222 = 0$$
$$x = \frac{-4 \pm \sqrt{16+888}}{2}$$
$$x = \frac{-4 \pm \sqrt{904}}{2}$$

$$\begin{array}{r} 888 \\ 16 \\ \hline 904 \end{array}$$

Só entendi deste jeito e dá errado

Vinícius G - **T4**

$$2{,}7x \cdot x = 481$$
$$2{,}7x^2 = 481$$
$$x^2 = 481 \div 2{,}7$$
$$x^2 = 17{,}81$$
$$x = 9{,}13$$
$$x = 11{,}151$$

Adriano I - **D2.7**

What is common to all the four solutions is the assumption that by modelling the problem with a numerical relationship and then numerically manipulating it is an acceptable method for solving the problem.

Summary of Findings and Conclusion

We think that the most important aspect in relation to this group of problems, is that it provides direct and clear illustration of different ways of modelling an "algebraic verbal problem," both algebraic and non-algebraic, particularly throwing light in the use of whole-part models, the superficial similarities and the deep differences between those models and algebraic ones.

It became clear that the choice of operations used in the solution process was mostly secondary to the modelling of the problem. In the case of algebraic solutions, it is the *arithmetical articulation*, as discussed in chapter 3, that informs the solution; in the case of whole-part solutions, it is the composition of the whole in terms of its parts—the *whole-part* articulation.

It was important to see, in Ticket[4] problems, the transformation of the more expensive ticket into "four tickets," ie, the application of the whole-part model independently from a "geometric" representation, indicating that those models are not simply a direct representation of the objects of the context; this suggests the possibility of the existence of a more general underlying model, in which case we would have a bigger obstacle to the development of an algebraic mode of thinking than if it were simply the case of totally contextualised solution, as an already established general model—even if not explicitly stated—would "compete" with the newly offered algebraic one. On the other hand, the teacher may take this to her or his advantage, by making the underlying *whole-part* model explicit, so it can be compared with algebraic models and the differences clearly established.

The fact that [2.7] problems are so more difficult if a *whole-part* model is used, can be understood in relation to the way in which the numbers involved are understood. Used with T&D problems, *whole-part* models impose a distinction between "the numbers that count the number of parts" and "the numbers that correspond to each part." Because the "unknown" parts are never dealt directly with, the notion of number that dominates in the model is that of *counting* number, and this clearly makes *whole-part* models not applicable at all to [2.7] situations. It is likely that teaching aiming at developing an awareness of the fact that, say,

2.7 x price per pound=price of 2.7 pounds

would significantly enhance the performance in [2.7] problems, but, as we have already indicated, the justification of such knowledge in terms of a decomposition of the decimal "coefficient" is far from immediately visible, so this seems to be an area to which anyone developing a teaching approach for the teaching of algebra has to pay careful attention.

Finally, the scripts in this section show ways in which, as we had indicated in the theoretical analysis of possible solutions, equations of the type

$$\mathbf{ax + bx = c}, \ \mathbf{a} \text{ and } \mathbf{b} \text{ positive integers}$$

can be modelled back into a *whole-part* model, but not if **a** or **b** are not integers; for the teacher or researcher, the fact that the model used can be completely hidden behind the use of "algebraic notation," indicates that it is not enough to *suppose* that the ability to solve equations of the type above imply the ability to solve the case with at least one of **a** and **b** non-integer.

We think that this is an extremely important result of our study, as it clarifies the inadequacy of "starting with examples with simple numbers" approach in the specific case of the types of equation involved in the solution of the problems in this section, but at the same time pointing out that a general problem exists in this respect, and that the underlying model has to be examined if we are to understand students' difficulties in learning algebra and in developing an algebraic mode of thinking.

4.3 SEESAW-SALE-SECRET NUMBER PROBLEMS

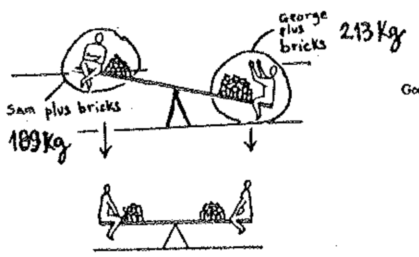
THE PROBLEMS

I am thinking of a "secret" number.
I will only tell you that ...

181 - (12 x secret no.) = 128 - (7 x secret no.)

The question is: Which is my secret number?
(Explain how you solved the problem and why you did it that way)

SN1 Problem

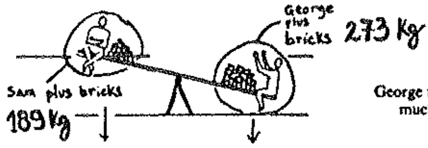


George throws away 11 bricks and Sam throws away 5 bricks.

Now they are balanced.

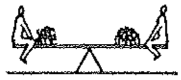
What is the weight of one brick ?
(Explain how you solved the problem and why you did it that way)

Seesaw 11-5 Problem



George throws away four times as much weight as Sam does.

Now they are balanced.



How many kilograms did George throw away? And Sam?
(Explain how you solved the problem and why you did it that way)

Seesaw 4x Problem

Maggie and Sandra went to a records sale.
Maggie took 67 pounds with her, and Sandra took 85 ponds with her (a lot of money!!).

Sandra bought 11 Lp's, and Maggie bought 5 Lp's.
As a result, when they left the shop both of them had the same amount of money.
What is the price of an Lp?

(Explain how you solved the problem and why you did it that way)

Sale 11-5 Problem

Maggie and Sandra went to a records sale.
Maggie took 67 pounds with her, and Sandra took 85 pounds with her (a lot of money!!).

Sandra spent four times as much money as Maggie spent.
As a result, when they left the shop both of them had the same amount of money.

How much did each of them spend in the sale?
(Explain how you solved the problem and why you did it that way)

Sale 4x Problem

GENERAL DESCRIPTION

This group of problems consisted of five problems, four of them contextualised (two contexts, Seesaw and Sale) and one "secret number" problem, where the problem condition is given in the form of a "syncopated" numerical equation.

Both Seesaw (**E**) and Sale (**A**) problems were presented in two distinct ways.

The first one gives the relationship between how much each of the two persons involved "threw away" (for **E** problems) or "spent" (for **A** problems) in terms of *number of pieces* ([11-5] problems). The second one gives that relationship in terms of *ratio* ([4x] problems).

Giving the relationship in terms of *number of pieces* sets the number of unknowns in the problems to only one, namely the weight of a brick or the price of an Lp (or a T-shirt, in the case of the Brazilian tests).

On the case of [4x] problems, on the other hand, they *primarily* involve two unknown quantities, linked by the given *ratio*, and the reduction into a problem with one unknown is a necessary step towards a correct solution of the problem, a step that involves a *substitution*.

The **SN1** problem was included in this group for the reasons already discussed in the introduction to this chapter.

On the Brazilian tests, **Sale** problems had numbers significantly larger than those on the English version, due to the necessity of adjusting the context to Brazilian prices. This may have discouraged trial-and-error solutions, but in any case trial-and-error solutions are not common in Brazilian classrooms, being in general explicitly characterised by the teachers as a "non-solution", and are not accepted by most teachers as a valid answer in a test. Although we insisted with the students that **any** method would be accepted, we expected a very low level of trial-and-error answers from the Brazilian groups — what actually happened — so the effect of larger numbers would be insignificant. We also chose to use "T-shirts" instead of "Lp's" because buying the former is a more usual activity for those students.

DISCUSSION OF POSSIBLE SOLUTIONS

Strictly speaking, [4x] problems are modelled algebraically by the set of equations

$$\begin{cases} \mathbf{a - x = b - y} \\ \mathbf{y = 4x} \end{cases}$$

while [11-5] problems are modelled algebraically by

$$a - 11x = b - 5x$$

From this point of view, [4x] problems are intrinsically more difficult than [11-5] problems.

However, it is possible that the given *ratio* is used to produce a *direct parts substitution* ("one lot and four lots") or a *direct numerical substitution* ("a number, four times a number"), thus reducing [4x] problems to the algebraic form

$$a - x = b - 4x$$

without going through the set of equations. From then on, both problems would be equally difficult from the algebraic point-of-view.

We expected non-algebraic solutions to fall into one of two main categories:

(i) a *qualitative analysis of the situation*, for example,

> " If George's side was heavier but now they are the same, it must be because **the amount George threw away in excess of what Sam did corresponded to the original difference between the two sides.**"

In this case, two subtractions would be performed in order to *evaluate* the original difference in weight and the number of *units* put away in excess, and then a division, in order to *evaluate* how much of the original difference corresponds to each *unit* thrown away in excess.

(ii) a *comparison of wholes* strategy, supported or not by a diagram (fig SSE 1)

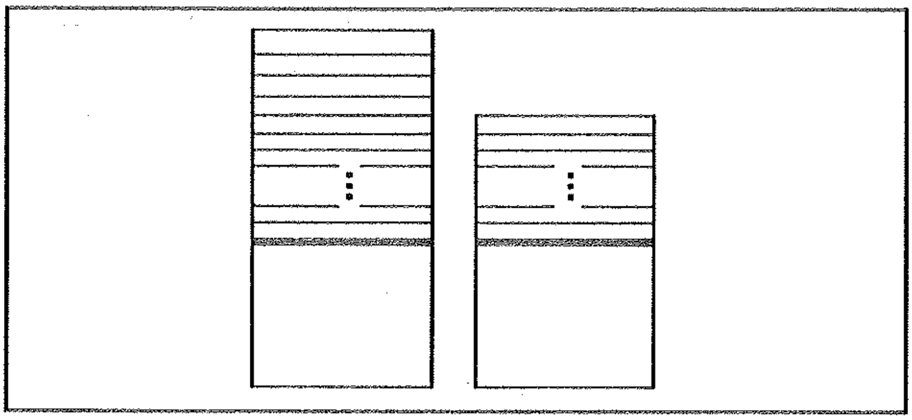


Fig SSE 1

Here two subtractions would also be performed, this time in order to *evaluate* the difference between the two wholes and the number of *units* "missing" on the smaller of the two wholes, and then a division, in order to *evaluate* how much of the difference corresponds to each *unit*.

The Secret Number (SN1) problem can be seen in three very distinct ways.

1) as *an equation in syncopated form*, in which case the numerical relationship could either be (1a) manipulated algebraically, or (1b) *modelled back* (for example, a scale-balance situation) and the resulting model manipulated to produce the answer .

2) as a *template*, providing a condition that has to be satisfied by the secret number but no information as to how to find it;

3) as *a compact description of a whole-part model situation*—eg, the one described some paragraphs above—that can be manipulated to find the required number. It is important to emphasise that this **does not** mean *modelling back* a numerical problem, but **actually seeing it that way from the beginning**. The subtraction signs are *literally* interpreted as "separating" or "removing" from the unequal wholes, an *action* that produces two new, equal, wholes.

There is a subtle but important difference between (1b) and (3). In (1b) the numerical relationship is recognised *as such*, although as a "by-product" of modelling a situation, and an effort is made to *model it back* into a setting where manipulation is possible; in (3), however, the arithmetical symbolism is never seen as such, once the expression involves an unknown number that cannot be used in calculations, and even worse, this number appears on both sides of the equality sign, completely removing any sight of a "result", and thus, any sight of "calculations". Instead, adding is seen as *joining*, subtraction as *disjointing or separating or taking away*, and multiplication as *grouping that many lots or parts.*

A study by John Mason (1982) reveals not only that symbols for arithmetical operations are easily used with this interpretation by young students, but also that when used in this way they might evoke properties different from those evoked by the arithmetical use, as in, for example, when trying to symbolise the Cuisinaire rods configuration in fig. SSE 2, where

3 x 3blacks and 2whites

can be consistently interpreted as

3(3blacks+2whites)

even in the absence of the original configuration (a correct interpretation in the context of the activity), but

3 x 3blacks + 2whites

might be interpreted, in the absence of the original configuration, as

(3 x 3blacks) + 2whites

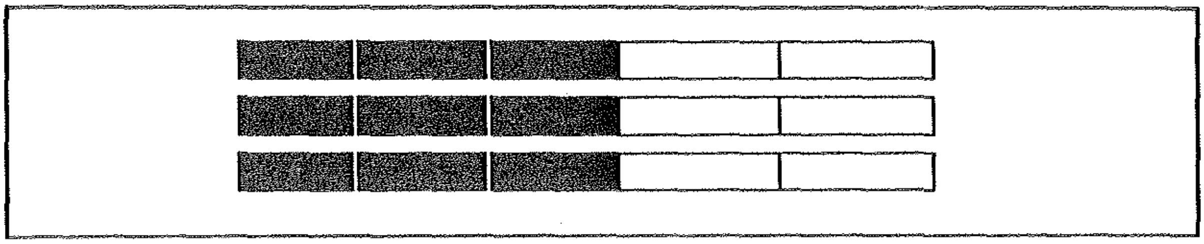


Fig SSE 2: configuration of rods to be described

The stronger bond produced by "and" is in correspondence to its use in normal speech, where in a phrase like "Sam and George's excellent performance!" the judgement is immediately seen as applying to both.

The use of non-algebraic models is bound by the necessity of maintaining a *dimensional homogeneity* when using addition and subtraction, ie, as far as the operations are used to *evaluate a total or a difference in measures*, the two operands must be seen as having the same *dimensional type*, once they are seen as *measures*. Algebraic models, on the other hand, avoids this concern *by introducing a homogeneity in numbers that can be sustained throughout exactly because of the* ***internalism*** *characteristic to thinking algebraically*. Dimensionality does not belong to the scope of *algebraic thinking*. This characteristic of the manipulation of non-algebraic models can serve, for example, to indicate the inadequacy of performing certain calculations (for example, on **E11-5** problems, the inadequacy of subtracting 11 (the number of bricks Sam threw away) from 273 (the initial weight on Sam's side)).

One aspect of algebraic and non-algebraic solutions is of special interest in relation to this group, because it is well recognisable in the range of different solutions to this group of problems.

In the general characterisation of our framework we have indicated that algebraic solutions are *analytical*. Moreover, we have seen that all the problems in this group can be correctly modelled by a numerical equation of the form

$$\mathbf{a - bx = c - dx}$$

Because the unknown appears on both sides of the equality sign, an algebraic solution to this equation cannot avoid *manipulating the unknown*, ie, adding or subtracting terms involving the unknown. But this is not an intrinsic characteristic of the relationship, it is rather a consequence of the *analytical* character of the algebraic method, of the need — so to speak — to express the unknown (required) number in terms of known numbers and operations on them.

We have also shown that the problems in this group, including **SN1** — and very similarly the above equation when **b** and **d** are whole numbers — can be modelled into a whole-part model, and that the manipulation of such model to produce the required number or measure completely avoids *manipulating the unknown* by producing successive *evaluations* of unknown measures from known ones, until one finally reaches a step where the unknown (required) measure is *evaluated.* Again, this is not a characteristic of the whole-part model itself, but of the *synthetical* character of *non-algebraic* methods.

Research on the solution of equations has indicated that there is a "didactic cut" in the passage from manipulating equations where the unknown appears on one side only of the equal sign to manipulating those where it appears on both sides, and that this cut corresponds to the "...need to operate on the unknown in the solution of [such] linear equations" (Gallardo, 1987).

Our analysis above indicates that the root of the difficulty with unknowns on both sides might lie on the fact that non-algebraic thinkers operate *synthetically* thus not operating with unknown values, ie, an important part of the strategy required to solve algebraically those equations does not fit into their normal, general framework. Also, it could be that the process of *translating back* a numerical equation with unknowns on both sides of the equal sign into a non-algebraic model is too difficult because of the complexity of the required models, and building some expertise on the process depends on a reasonable amount of experience. Nevertheless, students can be taught *translating back* skills (Gallardo, 1990).

Gallardo's example on page 44 (op. cit.) is particularly insightful, and we will examine it in some detail. It is about a student that had been taught to solve equations of the type

$$\mathbf{ax + b = cx + d \ , \ a>c \ , \ b<d, \ a,b,c,d>0}$$

by "...translating the equation's elements into a geometrical situation, where figures with equivalent areas were involved" (ibid.) (fig SSE 3).

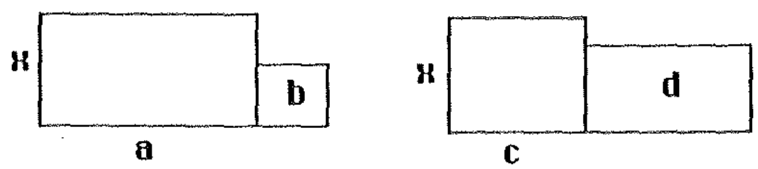


fig. SSE 3

When she had understood this model, she was then given the equation

$$9x + 33 = 5x - 17$$

which she modelled using the model taught with an "invention of her own": the subtraction of 17 was taken as meaning the *removal of a piece of the area equivalent to 5x.* (fig SSE 4)

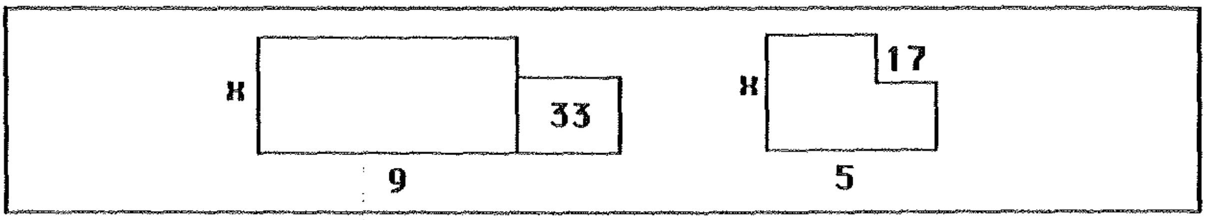


fig. SSE 4

The student manipulates this model to arrive at

$$4x + 33 + 17 = 0$$

corresponding to fig. SSE 5, and then a block occurs, because she is not willing to accept the negative solution.

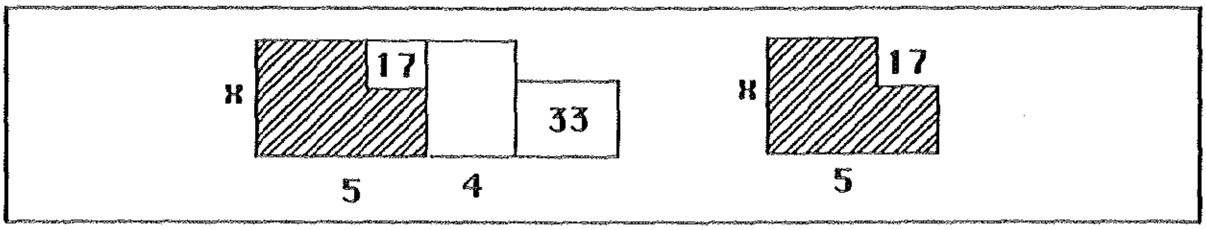


fig. SSE 5

This example is insightful, in the first place, because it suggests that the refusal to accept a negative answer is due to the fact that the "x" is representing the *measure* of a side in the figures, and thus can be but a positive number. In the second place, it shows the extent to which such solution is dependent on properties of the geometrical configuration, ie, the geometrical configuration is not just a *support diagram* to help to keep track of a reasoning that is "in essence" identical to the one behind an algebraic solution. Finally, this example supports our suggestion that the process of *translating back* is far from simple and straightforward, as finding a similar geometrical configuration to model and solve an equation like

$$173 - 5x = 265 - 11x$$

would certainly involve either a reasonable amount of experience with such models, having being taught the configuration as a "solution formula", or a high degree of ingenuity[13].

* * *

On the basis of our analysis of the problems, we hypothesized that:

A) [4x] problems might be more difficult to solve than [11-5] problems for a student using a non-algebraic approach, because [11-5] problems provide *objects* (bricks or Lp's) that can be immediately seen as *parts*, while on the case of [4x] problems one has first to establish a *unit* (more easily, how much Sam threw away or how much Maggie spent) to be then manipulated as a *part* and to represent the "4 times" as "4 parts" or "4 lots";

B) [4x] problems might be easier to solve if an algebraic approach is used rather than a non-algebraic one, because the "4 times as much" statement would suggest within a *Numerical Semantical Field* — by suggesting a multiplication — the correct "unknown, 4 times the unknown" structure; this approach reduces the difficulty of having to establish a *unit*, once seeing the "4 times as much" — *times* indicating a *ratio* — as meaning "4 times the other amount" — *times* indicating multiplication — immediately entails the "other amount" that is to be multiplied as an *object* (multiplication requiring two numbers to be performed). The predominant use of an algebraic approach within a group of students would thus reduce the difference between the facility levels for [11-5] and [4x] corresponding problems.

C) **SN1** problems would be extremely difficult to solve using a non-algebraic approach.

GENERAL DATA ANALYSIS

One aspect of the data is helpful in understanding other aspects on the data, so we examine it first.

For both Brazilian groups the **SN1** problem had the highest level of facility among the problems in this group (43% for AH7 and 88% for AH8), all but one of the correct solutions employing equations. On the other hand, for both English groups the **SN1**

[13]The degree to which this is true can be easily verified by trying to produce such configuration and to solve the equation using it. It was not immediately that I found a way out of it myself.

problem had the lowest facility level among the problems in this group (4% for FM2 and 15% for FM3); four of the seven correct answers employed equations.

Those numbers are a direct indication of the extent to which Brazilian pupils dealt better with equations than their English counterparts, once eventual difficulties with modelling the problem onto an equation are almost reduced to none. More important here, however, is the fact that solving **SN1** problems depended so heavily on the use of equations.

Only 4 students on the combined FM2-FM3 group (75 students solving **SN1** altogether) tried to use an equation with **SN1** and failed to solve it correctly. Together with the very low level of success on **SN1** that suggests that students on the FM2-FM3 group were predominantly trying to use non-algebraic methods to solve **SN1** problems.

Another aspect of interest arising from the data is the use of equations on corresponding [11-5] and [4x] problems. In almost all cases — the exception being **A11-5** and **A4x** for FM3, where the use of equations was nil for both problems — the percentage of correct solutions using equations is higher for [4x] than for [11-5] problems[14]. This indicates that algebraic solutions do belong to a *Semantical Field* where numerical relationships are meaningful by themselves, as the suggestion of the multiplication seems to be the factor that triggered the choice of an algebraic solution.

More support for this interpretation can be drawn from the fact that on the AH7 group the bulk of the correct answers to [11-5] problems came from non-equation solutions but **all** the correct solutions to [4x] problems used equations. Algebra is systematically introduced only on the 7th grade of Brazilian schools, usually later on the first half of the academic year; thus, seventh graders can be considered *well informed* and somewhat *skilful* in solving equations, but not yet deeply committed to using equations whenever they are given a verbal "algebraic" problem. This can be also seen in the fact that in all of the four contextualised problems, most of the incorrect solutions on the AH7 group do not attempt to use an equation and most of the incorrect solutions on the AH8 group **do** represent a mistaken use of equations. This suggests that for the Brazilian 7th graders the "default" approach is non-algebraic, and for the 8th graders it is an algebraic one, namely the use of equations.

[14]This difference is significant on the Brazilian groups, although it is not significant on the English groups due to the very low level of correct answers using equations.

The use of algebraic methods resulted — as we have predicted — in very similar facility levels for three out of four pairs of corresponding [11-5] and [4x] problems on the Brazilian groups, while on the English groups [11-5] problems were always significantly easier than the corresponding [4x] problems.

On the Brazilian groups **SN1** has a high facility level, and the lower levels of correct answers to the four contextualised problems indicate *difficulties with modelling them with an equation*, ie, with establishing a correct arithmetical relationship; this is even more evident as we look at the percentages of **incorrect** solutions involving equations at AH8, that "by design" (curriculum) is bound to use equations more than AH7. On the other hand, on the English groups **SN1** has a low facility level, and the differences between corresponding contextualised problems reflect *difficulties in seeing meaningful relationships between the elements in the context of the problems.*

The former difficulty might be seen as having a greater degree of complexity, as one would have to make sense of the structure of the given situation and *then* transform it into a numerical-arithmetical problem. However, the *mode of thinking* in which one is operating is of substantial importance in determining for a given problem the degree of difficulty in understanding the structure of a problem. The fact that a person is aiming at transforming a contextualised problem into a numerical-arithmetical one may be, as we saw in relation to [4x] problems, of great help in making sense of a structure for the problem, which shows that difficulties with the algebraic approach do not represent the simple accumulation of the *numerical* difficulties on the top of other difficulties in understanding the structure of the problem.

STUDENTS' SOLUTIONS

The SN1 problem

All of the 43 **OKEQT** solutions by Brazilian students (of a total of 71 students presented with the question) used standard algebraic symbolism while the three **OKEQT** solutions by English students (out of 75) employed "secret no", "sn" or "?". In itself this suggests that the use of a special form of symbolism, rather than syncopation or the "iconic" interrogation mark might become a significant factor in establishing equations as recognisable—and thus acceptable and capable of being manipulated—mathematical objects. This suggestion is supported by a number of explanations presented with the solutions (Bartira G, AH7; Ana B, AH8; Eurico G, AH8):

nº secreto = x

$181 - (12 \times x) = 128 - (7 \times x)$

$181 - 12x = 128 - 7x$

$-12x + 7x = 128 - 181$

$(-1).\ -5x = -53.(-1)$

$5x = 53 \qquad x = \frac{53}{5}$

$x = \frac{53}{5}$

Quando eu falo que o nº secreto é x. É porque x pode ser um nº qualquer. É incógnita.

Bartira G, AH7: "When I say that the secret number is x, it is because x can be any number. It is [the] unknown."

$181 - (12x) = 128 - (7x)$

$181 - 12x = 128 - 7x$

$181 - 128 - 12x + 7x = 0$

$53 - 5x = 0 \rightarrow -5x = -53$

$5x = 53$

$\boxed{x = \frac{53}{5} \rightarrow x = 10,6}$

Eu substitui o "nº secreto" que está na dica, por x e depois transformei a dica em uma equação e resolvi-a. Até ~~[illegible]~~ descobrir o x.

O nº secreto é 10,6

Ana B, AH8: "I replaced the "secret no." that is in the *hint* by x and then *transformed the hint into an equation* and solved it until I found out the x." (our italics)

$181 - 12x = 128 - 7x$

$5x = 181 - 128$

$x = \frac{53}{5}$

R: O número secreto é $\frac{53}{5}$

Eu peguei a fórmula dada e substitui ~~[illegible]~~ o nº secreto por uma incógnita, após fazer isso passei as incógnitas p/ um lado e os números p/ o outro, aí foi só concluir

Eurico G, AH8: "I took the given formula and replaced the secret no. by an unknown, after this I moved the unknowns to one side and the numbers to the other, then it was just a matter of completing [the solution]."

In 19 out of the 43 **OKEQT** solutions by Brazilian students, an intermediate form is produced between the problem's statement and the equation in its standard form, putting **12x** and **7x** or **12** x **x** and **7** x **x** in brackets (as Bartira G, AH7, script already shown, did), an aspect that also supports that suggestion.

In 23 **OKEQT** solutions by Brazilian students, the following line appeared:

$$-5x = -53$$

instances showing that in algebraic solutions the meaningfulness of each expression produced is related only to the perceived correctness of the process that produced it, ie, the *internalism* of thinking algebraically.

A variety of algebraic techniques appeared on the **OKEQT** scripts:

(i) multiplying both sides by (-1) to get rid of the negative signs (Cláudia F, AH7) or to transform the side of the equation containing terms in the unknown into a more appropriate form (Andrea M, AH8);

$181-(12 \times x) = 128-(7 \times x)$
$181-12x = 128-7x$
$-12x+7x = 128-181$
$-5x = -53$
$-1.(-5x) = -1.(-53)$
$5x = 53$
$x = \frac{53}{5}$ $x = 10,6$

Resposta. O número "secreto" é 10,6

EXPLICAÇÃO: Tanto você fazendo a operação 1 ou a operação 2, o resultado será o mesmo.

Claudia F, AH7

$181-12n = 128-7n$
$-12n+7n = 128-181$ $(\times -1)$
$12n-7n = -128+181$
$5n = 53$
$n = 10,6$

como no enunciado já existia uma expressão era só trocar a palavra número secreto por uma variável qualquer e separar parte literal e parte numérica.

Andrea M, AH8

(ii) directly performing the division (-53)÷(-5), without first performing the step described on the previous item (Ernesto K, AH7);

$$
\begin{array}{ll}
181 - 12x = 128 - 7x & \begin{array}{r} 181 \\ -\ 128 \\ \hline 053 \end{array} \\
-12x + 7x = -181 + 128 & \\
-5x = -53 & \\
x = -53 : (-5) & 53 \,|\, 5 \quad 10,6 \\
x = 10,6 & 030 \\
 & 0
\end{array}
$$

Ernesto K, AH7

(iii) transforming the equation into a standard form (ax + b = 0, Ana B, AH8, script already shown on this section), (ax + b = cx + d, Robert M, FM3);

181 - (12 x Secret no = 128 - (7x secret no)
181 = 128 - (7x secret no) + 12 x sn)
181 + (7 x sn) = 128 + (12 x Sn)
53 + (7 x Sn) = 12 x Sn.
53 = 5 x Sn
53/5 = SN.
SN = 10.5

Robert M, FM3

(iv) expressing the answer both as a fraction or as a decimal number;

One solution is of particular interest (Nick A, FM3). Apart from the use of "?" for the unknown, it seems to present us with a mixed solution. The first step,

181 - 12x? = 128 - 7?

181 - 5? = 128

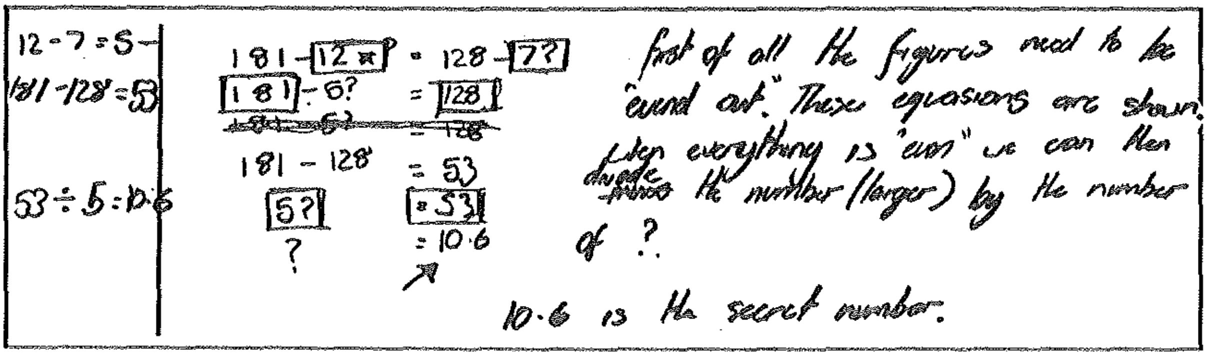
could be seen as the result of an algebraic manipulation. The second step, however,

181 - 5? = 128

181 - 128 = 53

5? = 53

seems to be based on a *whole-part* modelling of 181 - 5? = 128, once no intermediate step is provided except the evaluation of 181-128, and the transformation seems to be a direct one. Whether the first step was also based on a non-algebraic model, nothing can be concluded.


12-7=5
181-128=53
53÷5=10.6
181 - 12 ? = 128 - 7?
181 - 5? = 128
181 - 128 = 53
5? =53
? = 10.6
first of all the figures need to be "evend out." These equasions are shown. When everything is "even" we can then divide the number (larger) by the number of ?.
10.6 is the secret number.

Nick A, FM3

From all four groups (a total of 146 students presented with the question) there were only five **OKCALC** solutions to **SN1**. This immediately indicates that to model **SN1** into a non-algebraic model was a very hard task for those not able to use an algebraic one for whatever reason.

Of the five **OKCALC** solutions, Elizabeth W's (FM3) was certainly the most peculiar. First, because she **does** produce the right number, using the most direct calculations possible, only to "conclude" that — for some unexplained reason — 10.6 is **not** the secret number. Second, for the *rationale* to her choices of subtractions ("181 is bigger than 121 and 12 is bigger than 7"). However, it is difficult to see why she chose to divide 53 by 5, and not to perform some other operation. The numerical preference "divide the bigger by the smaller" cannot provide a justification for the choice of a division itself, and we are led to believe that she did have the insight of an underlying non-algebraic model, and she so expressed herself because she was not able to make the model explicit — even to herself. Another interesting aspect is that she never thought of **trying** the 10.6 she thus obtained to see if it "worked", saying instead that she would use a **trial**-and-error approach.

At first I thought I would answer this with "trial & error," but I have had a better idea (if it works!). 181 is bigger than 128, and 12 is bigger than 7, so I got this idea: 181 − 128 = $\boxed{53}$ 12 − 7 = $\boxed{5}$. 53 ÷ 5 = 10.6. No, that was a bad idea − 10.6 is not the secret number. I will use trial + error. 12 × 5 = 60

Elizabeth W, FM3

Two of the remaining four **OKCALC** solutions (Fabiana M, AH7; Gareth A, FM2) do not provide us with information enough to decide whether they represent non-symbolic solutions of an equation. Even if they are not, this is probably as close to it as we will get, once Gareth actually produces a standard equation (replacing "secret no." by "**x**") and Fabiana says "to know the difference between known numbers and between unknown numbers and divide them". Another possibility would be, as we have already seen, to reason in a manner similar to that described as possible non-algebraic solutions to the contextualised problems, only this time reasoning with the numbers themselves:

> "The amount of secret nos. that is taken in excess from the left-hand side must be the difference between 181 and 128", etc..

and this seems to be exactly the model used by Joe V (FM3) and Jacob B (FM3).

(181 − 128) : (12 − 7)

53 : 5

$\boxed{10\frac{3}{5}}$

$\begin{array}{r} 181 \\ -128 \\ \hline 053 \end{array}$ $\begin{array}{r} 12 \\ -7 \\ \hline 5 \end{array}$

saber a diferença dos nºs conhecidos e dos nºs des conhecidos e dividi-los.

Resp: o nº é $\boxed{10\frac{3}{5}}$

Fabiana M, AH7

181-(12x X)=128(7xX)

181-128 = 53

12÷7 = 5

5/53 = 10 3/5

Gareth A, FM2

10.6
5)53
5
30

12x10.6=127.2
7x10.6=74.2

181.0 128.0
-127.2 -74.2
53.8 53.8

First I need to find what 12 x secret no = etc so I take 128 from 181.
181
-128
53 ÷ 12-7 (numbers than you times secret numbers by)
53 ÷ 5 = $\boxed{10.6}$
now I check that this is wrongtt by dong these sums 181-10.6x12=53.8 → 128-7x10.6=53.8

Joe V, FM3

The difference between 128 and 181 is 53, and as *12x is 5 more than *7x, I must find out how mony 5's go into 53, the answer is 10.6, this is the secret no. secret no. = 10.6

181-(12x10.6) = 128-(7x10.6)

181-(127.2) = 128-(74.2)

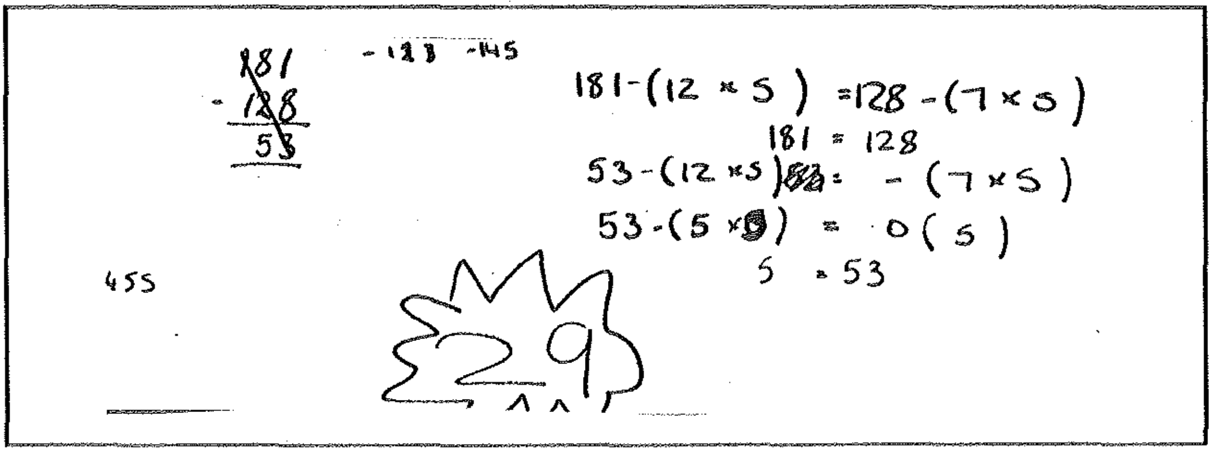
53.8 = 53.8

Jacob B, FM3

There were altogether 11 **WEQT** solutions. In three of them the original equation was correctly manipulated up to a point, and then the solution process was halted. In one case (Russell P, FM3) the difficulty came when he reached the equation

$$53 - (5 \times s) = 0\ (s)$$

to conclude that s=53. It appears that the difficulty lied in perceiving that 0s=0.



Russel P, FM3

One student, Shelley S (FM2, script not shown), replaced "secret no." by "x" but failed to go any further.

Jack D (FM3) tried to apply a *scale-balance analogy*. It is interesting that he stopped (and crossed out his previous efforts) when he reached (through a sequence of mistaken steps) the equation

$$\mathbf{53 - (5 \times SN1) = 0}$$

but it is equally interesting to observe that the use of such model produced two mistakes that are clearly associated with treating the problem using the *scale-balance analogy*:

(i) the analogy treats the unknown number as the unknown weight of an object; although the minus sign is kept on the left-hand side, probably meaning "removal", a "negative" amount of objects or "removing 7 objects from nothing" does not make sense in the *Semantical Field* of the *scale-balance analogy*. Thus, the minus sign is simply dropped.

(ii) on the second step, he says "**take off** 7 from each side", where the correct algebraic strategy would be "**add** 7 [xSN] to each side" or at least — given the equation on which he was operating — "**add** 12 [xSN] to each side". That by using this incorrect strategy he produces the transformation

$$\mathbf{53 - (12 \times SN1) = (7 \times SN1)}$$

to

$$\mathbf{53 - (5 \times SN1) = 0}$$

is enough evidence that the subtractions were thoroughly ignored by being meaningless in this *Semantical Field*.

Balance the scales.
181 - 128 = 53.
So 53 (12 x 5N) = (7 x 5N)
Then take off 7 from each side.
So 53 - (5 x 5N) = 0.

Jack D, FM3

There is an important point to be discussed here. The *scale-balance analogy* has been one of the most popular didactic artifacts used to teach the solution of linear equations. Let us analyse the use of such analogy to model equations of the form

$$\mathbf{a + bx = c + dx \ \ , abcd \neq 0}$$

for various sets of conditions for the parameters **a**, **b**, **c**, **d**.

- **a>c, b<d, b and d positive integers** (eg, 100 + 10x = 80 + 15x)
  On such cases, the analogy thoroughly applies; the plus sign is understood as *conjoining*, and thus there is a definite correspondence between the "taking off weights" strategy on the scale-balance model and the "subtracting a quantity of x's" on the algebraic model, and also division corresponding to *evaluating* a sharing action.
- **a>c, b>d, b and d positive numbers** (eg, 100 + 15x = 80 + 10x)
  On this case the analogy simply does not apply: it is not possible to put more objects on the side that is already heavier and make it balanced. Unless, of course, that the objects have *negative weight*, an impossibility within the *Semantical Field* of the scale-balance.
- **a>c, b<d, b and d positive non-integers** (eg,..........................
  100 + 3.4x = 80 + 7.8x)
  The difficulties arising here because of the decimal numbers were analysed in depth when we discussed the *Ticket and Driving* problems. The meaning of "3.4 objects" is not at all natural within the *Semantical Field* of the scale-balance, and an extension that makes it meaningful is not easy to grasp.
- **a>c, b>d, b and d negative integers** (eg, 100 - 15x = 80 - 10x)
  As analysed with Jack D's script.

It is not necessary to go any further. One obvious problem with the *scale-balance analogy* is the limitation imposed on the coefficients of the unknown and on the sign of the unknown itself. Certainly more important, the variety of strategies required to use this analogy across equations with different sets of conditions for the parameters is in clear contrast with the fairly reduced set of principles and strategies used with an algebraic model. As a consequence, the *scale-balance analogy* is inadequate not only for very quickly becoming a complex net of what are in effect different models, but also for not fostering a frame of mind adequate for the development of an *algebraic mode of thinking.*

In the remaining 6 **WEQT** solutions, the errors are always in the manipulation of the equations, as in Lilian P's (AH8) script. Those types of errors are well documented by research and in teaching practice.

$$181 - (12 \cdot x) = 128 - (7 \times x)$$
$$181 - 12x = 128 - 7x$$
$$53 - 19x = 0$$
$$-19x = -53$$
$$(-1)x = \frac{-53}{19} \rightarrow x = \frac{53}{19}$$

Lilian P, AH8

The 27 **WCALC** attempts divide naturally into two groups. In one of the groups (21 scripts), a subtraction **181-128** was always attempted. It is not possible to decide from the scripts whether those students were producing a first step in the solution of an equation of the type

$$\mathbf{181 - 12x = 128}$$

temporarily putting away the **-7x** term, or just "taking away the smaller from the greater". In any case, it is clear that manipulating the unknown or even its coefficients in a meaningful way presented a much greater degree of difficulty. Some attempts proceeded by dividing 53 — the result of the subtraction — by 12, which again appears to be the result of dealing with the incomplete equation above; some others multiplied 53 by 12 or by 7, clearly for not grasping the structure of the equation. Two students in this group (one of them Ian C, FM3) produced the subtraction **12-7** but failed to use this information correctly, which again shows a lack of grasp of the structure of the equation.

$\cancel{181 - 128 = 53}$

$181 - 128 = 53$

$12 - 7 = 5$ $\qquad$ $53 - 5 = 48$

Ian C, FM3

All but one of the remaining students in the **WCALC** category seem to be merely attempting to produce a "sensible answer" by trying different combinations of operations with the numbers given. Alessandra S's (AH8) attempt, however, exhibits some intention to manipulate numerical equalities but no sense of how to do it; it is interesting that she takes the 7=7 equality as signaling the end of the process, clearly of formal meaning only.

$181 - 7 = 128 - 7$ $\qquad$ O nº secreto é 7

$174 = 121$

$181 - 174 = 128 - 121$

$7 = 7$

Alessandra O, AH8

The Seesaw 11-5 problem

Only 5 out of 77 students presented with this problem correctly used an equation to solve the problem (**OKEQT** solutions); one of them had to be categorised as an incorrect answer once he simply erased his correct solution (which, of course, still remained visible). Those solutions do not provide much additional information on the solution of equations. However, in one script (Andrea M, AH8) we have a quite clear description of her process of solution.

$189 - 5x = 273 - 11x$
$189 - 5x - 273 + 11x = 0$
$6x = 273 - 189$
$6x = 84$
$x = 14$

o peso do tijolo é x, e para formar uma igualdade precisava estar os dois pesos equivalentes, como essa equivalência foi dada foi só montar as duas contas de subtração.
O resto é o mesmo processo do isolamento do x, fazendo a operação inversa.

Andrea M, AH8

(i) the brick' weight is **x**...

(ii) and to form an equality we would have to have both weights equivalent...

(iii) as this equivalence was given...

(iv) I only had to assemble the two subtraction sums.

(v) the rest is just the process of isolating **x**, doing the inverse operation.

From considerations involving characteristics particular to the problem's context—namely, that seesaws are balanced only when the weight on both sides are the same—she moves into a numerical-arithmetical context, and then solves the equation. This is, thus, an exemplary case of *algebraic thinking* "in action."

The **OKCALC** solutions are roughly equally divided between two solving strategies:

i) *qualitative analysis of the situation*, as we have already described at the beginning of the section on this group of problems (Tarek S, AH7, provides a clear written explanation)

$$\begin{array}{r} 273 \\ -189 \\ \hline 084 \end{array} \qquad 84 \,|\, \underline{6} \quad 24 \;\; 14$$

Jogando-se 11 tijolos de um lado e 5 do outro, a diferença fica = a diferença do peso. Então é só dividir o peso pelo nº da diferença de tijolos
14
Cada tijolo pesa quilos

Tarek S, AH7: "Throwing away 11 bricks from one side and 5 from the other, the difference becomes [equal] to the difference in weight. Then, one has only to divide the weight by the number of the difference of bricks"

(ii) *hypothetical manipulation of the context* (Bridget S, FM3). This strategy is different from (i), as it actually transforms the problem into another one. The fact that the subtraction **11-5** still had to be performed is not as relevant here as the importance — in finding a solution — of the new image generated.

213 - 189 = 24kg ÷ 6 = 4kg   1 brick = 4kg
If George throws away only 6 than he is equal
as Sam if sam doesn't throw away any so
you subtract 189kg from 213kg and divide by 6 bricks

Bridget S, FM3

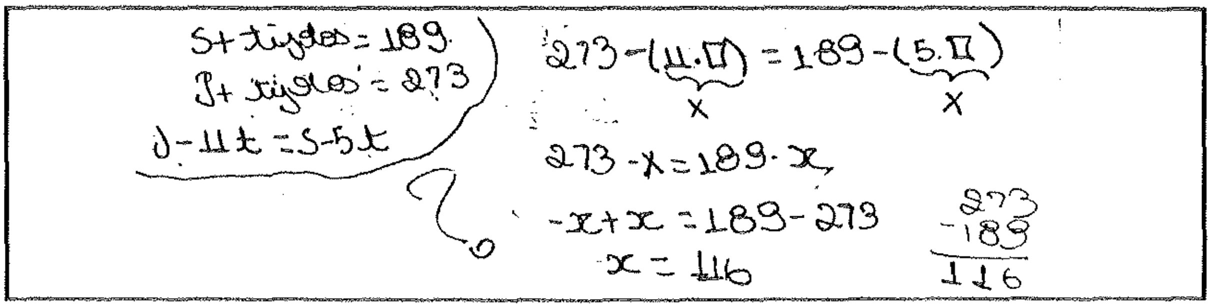
In no solution a diagram like the one we provided with the *comparison of wholes* strategy was produced, and the fact that all **OKCALC** solutions mention "weight" or "bricks" or both in association with the numbers produced strongly indicates that it was not used "in the background" either[15].

In all **WEQT** attempts we could identify mistakes deriving from a very loose use of the algebraic notation.

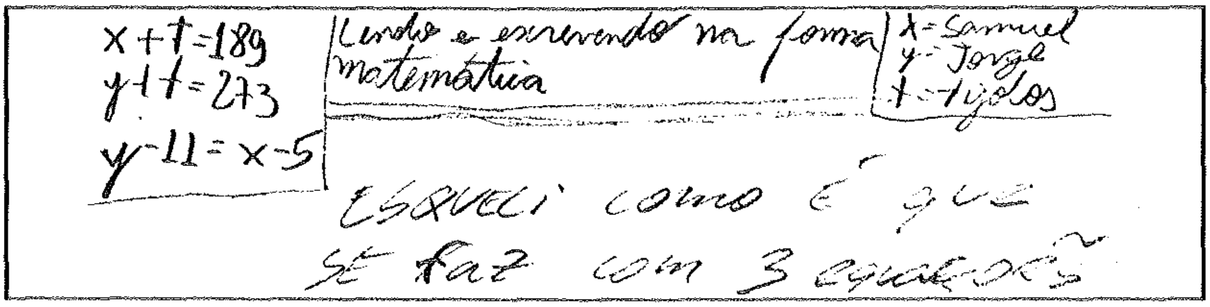
One student (Fabiola, AH7), first produced a syncopated translation of the problem (left upper corner), that apparently served as the basis for writing the (correct) equation on the first line — using a box for the weight of a brick. She then replaces the two occurrences of the box with their coefficients, by **x**. The reason is not clear at all, and this is the step that produces the critical mistake. This script is interesting for bringing together three different uses of notation: descriptive and both standard and non-standard algebraic and the urge to use **x** to make the expression on the first line into a recognisable equation is certainly related to the same aspects we discussed in relation to **OKEQT** solutions to

[15]We want to emphasise that we have already commented on page ... on the distinction between "there is in any case a whole-part structure manipulation" and "a comparison of wholes strategy is used".
We think it would not be an useful approach here, to consider that some form of abstract comparison of wholes structure was "actually" used "in the background". The crucial distinction between the *comparison of wholes* strategy as we described it, and the two strategies used by the students, is that the problem is transposed to another — in this case, more general — embodiment, one where the notion of measure is used in a different way.

SN1. Another good example of a descriptive use of literal notation is found in Marcel S's (AH8) script, who also adds: "Reading and writing in *mathematical form*" (top, our emphasis) and "I forgot how to do it with 3 *equations* [sic]" (bottom)[16,17].

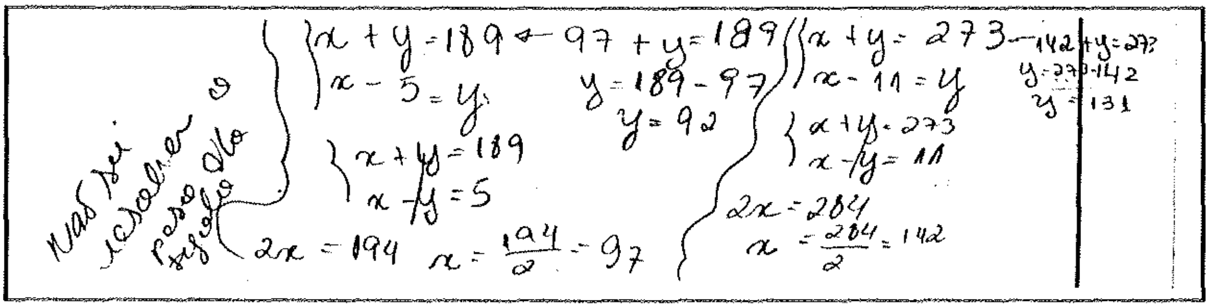


Fabiola, AH7



Marcel S, AH8

Other mistaken solutions show a combination of loose and incorrect use of notation with poor understanding of the elements and structure of the problem (Marina F, AH8).



Marina F, AH8

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[16]In Portuguese, "tijolos" stands for "bricks".
[17]Although the expressions are clearly descriptive — for example, by the use of **t** ("tijolos") for both amounts — the literal notation leads the student to see them as *equations*. The usual Brazilian teaching practice puts much emphasis on "doing with letters" on the one side and "algebra" and "equations" on the other.

Many of the **WCALC** solutions (9 out of 16) are *contextwise homogeneous*, ie, the calculations produced always involve pairs of numbers that measure the same kind of thing (eg, weight). Those solutions were either incomplete (simply subtracted the smaller weight from the greater), considered that the difference in weight had to be shared between the **total** number of bricks involved (Clare B, FM3, a script that illustrates well *contextwise homogeneous* solutions), or considered that the **total weight** had to be shared between the total number of bricks. Of the remaining **WCALC** solutions, three used the representation

$$\mathbf{189 - 5 = 273 - 11}$$

which seems to be a mere (incomplete) syncopation of the problem's statement. In two of the cases it resulted in the focus of solution being totally diverted to the calculations involved, with no regard for the structure of the problem (Ana F, AH8). The other student did not go any further, and this suggests that she kept the awareness that it was only an incomplete syncopation.

213 − 11 = 202	~~189 − 21~~
189 − 5 = 184	213 − 189 = 24 = the difference between them
202 − 184 = 29	11 + 5 = 16 total number of bricks
11 + 5 = 16	24 − 16 = 1.5 kg
29 ÷ 16	

Clare B, FM3

273 − 11 = 189 − 5	5 = 189 − 5	5 = 184
262 = 184	J = 273 − 11	J = 262
		262 − 184 = 78
		78/2
		39

Ana F, AH8

The only aspect of interest on **T&E** solutions, is that none of the students actually wrote down numerical-arithmetical expressions involving the variable to be tested that would serve as a *template* for testing the "guesses". As we said before, **T&E** solutions are in a sense closer to algebraic solutions than non-algebraic solutions, both because the original problem is transformed into a numerical-arithmetical one and because the notion of *variable* is involved, even if in a rudimentary form; nevertheless, the lack of a representation of the *template* makes it difficult for the students to go beyond the trial-and-error process and to perceive the numerical-arithmetical equality as an *object* that could be directly manipulated to produce the required number[18]. That those students in our study had the *template* represented in some internal form, is out of doubt; Sanjay (FM3) actually writes down an "algebraic" version of the *template* to illustrate the condition that his guess would have to satisfy, and immediately substitutes a value to show it is the correct answer. The fact that both the *template* and the "confirmation" calculations have in fact the subtractions inverted — but to produce correct results — shows the extent to which the notation is merely descriptive.

Each brick weighs 4 kg   I found answer by ~~finding~~ estimating a number for the brick & carry out the calculation

$11a - 213 \quad 5a - 189 = 11 \times 4 - 213 \quad 5 \times 4 - 189$

$= 169 \qquad = 169$

Sanjay, FM3

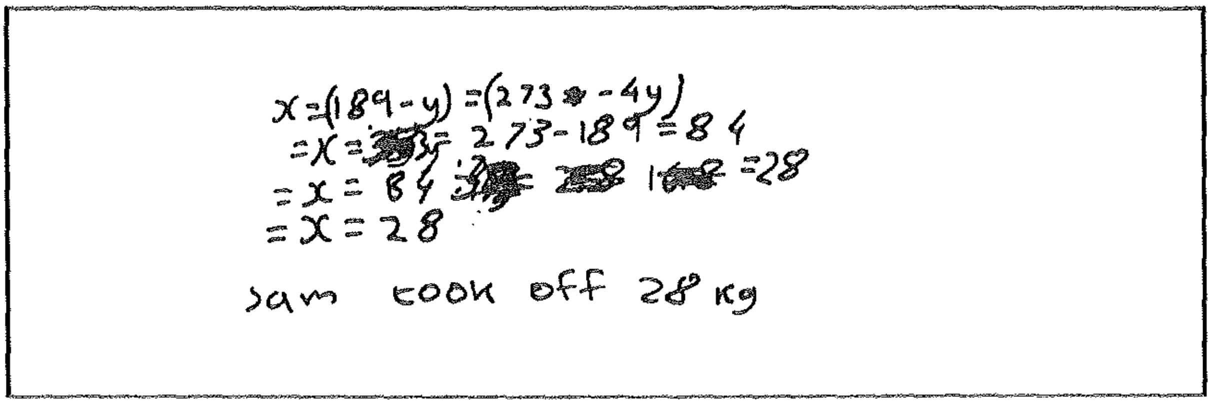
<u>The Seesaw 4x problem</u>

The **OKEQT** solutions to the **E4x** problem do not add much to what we have already said about **OKEQT** solutions in the analysis of the previous two problems in this group. One aspect only is worth mention, that of the three **OKEQT** solutions coming from English groups, in only one the use of symbolism is totally standard[19]. The other two solutions use algebraic notation in much less standard ways. Sukhpal (FM3) uses an extra — descriptive — **x** to reaffirm to herself that both sides will come to a same total, while Keith W (FM3) keeps the multiplication sign with the coefficients of the unknown and mixes lines with an equation with lines with numerical calculations only; his solution does

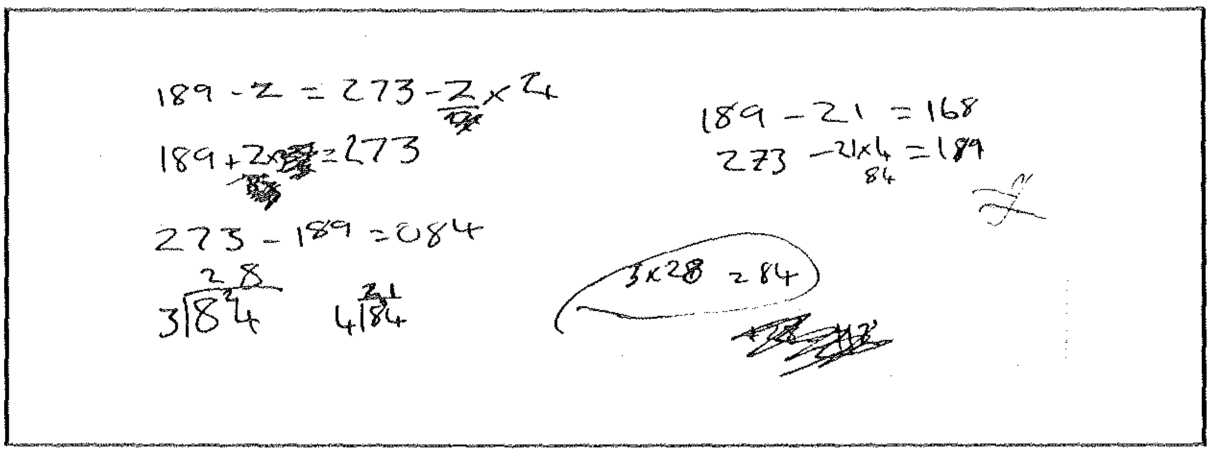
[18]In a study by C. Kieran (mentioned in Kieran, 1988), "those [pupils] who preferred substitution viewed the letter in an equation as representing a number in a balanced equality relationship; those who preferred inversing viewed the letter as having no meaning until its value was found by means of certain transposing operations."

[19]Actually, this student was a visitor from Bulgaria, where, judging by the tradition of the pedagogy of Eastern Europe countries, much attention is paid to the formal aspect of algebraic symbolism.

not reach a formal end, and one has to assume its correctness from the encircled **3 x 28 = 84** expression at the bottom.



Sukhpal FM3



Keith W, FM3

All **WEQT** solutions come from Brazilian students, and there is always an initial mistake in setting the equation. The one worth noting is Celia R's (AH7), because her main mistake (reversing the written form of the subtractions) is also seen on purely arithmetical contexts[20].

[20]In this case, **x-189** could be representing "take **x** from [-] **189**", a literal, non-mathematical translation of the textual structure of the problem. From this and other examples, one should be aware that the using the notion of *translation* to describe the process of transforming a contextualised problem into a numerical-arithmetical equations might be a didactic mistake, as much as it involves the false notion that "it is the same thing, only said in a different *language*". Of course, the notion that "algebra is a *language*", itself mistaken, is in the **root** of such misleading statement.

$S = x - 189 = 56,6$ $\quad 5x = 462$

$J = 4x - 273 =$ $\quad x = 462 : 5$

$x = 92,4$

462 | 5
12 92,4
2

$$\begin{array}{r} 189 \\ +273 \\ \hline 462 \end{array}$$

ñ Sei

Celia R, AH7

Significantly, only two **OKCALC** solution (out of 77 scripts) were produced, confirming our prediction that establishing a *unit* that could be manipulated as a *part* would be a major difficulty for students not using an algebraic model. The two scripts show only the calculations, and present no verbal explanation of the process of establishing the *unit.*

WCALC solutions provide an even stronger confirmation of our prediction. 20 out of 24 **WCALC** attempts simply ignored that there was 1 part (Sam's) to be considered. In 9 of those solutions the students gave the difference between the weights as the answer (James O, FM2) and in 10 of them the **4** is used to *divide* or *share* the difference between the weights (Helen C, FM2). Four students did considered Sam's one part, but in three of those cases they also considered that the amount to be shared into **5** was the **total weight,** and not the difference (Fabio P, AH7). It seems that because they were thinking of **total weight** the **total amount put away** had to be considered, and this led them to the **5** divisor.

George throw away 84 kg
I did it that way because it was
the first one that I thought
of.

James O, FM2

George threw away 84 kg and sam threw 21 kg away
I subtracted George's weight with Sam's weight and I got 84
so i divided 84 by 4 and Got 21
I did it this way because of the way the problem was set up

Helen C, FM2

2,7,3   24 |15
1.8,9   34   16,8
084   40   ×4 = 67,2

Jorge = 67,2
Samuel = 16,8 k
Com a diferença entre os dois eu pegu o quantas vezes eles tiram
e dividi pela diferença e resulta o quanto Samuel tirou e o do Jorge multiplicar por 4.

Fabio P,AH7

"With the difference between the two, I took how many times they took away and divided by the difference [sic] and the result [is] how much Samuel took away and for Jorge multiply by four."

It is clear that the **E4x** statement did not easily provide *parts which can be manipulated* for the weights wasted by Sam and by George, and the fact that this caused major difficulties for those students strongly suggests that the models they were using depended heavily on that kind of *object.*

<u>The Sale 11-5 problem</u>

One characteristic aspect of the algebraic method appears in three of the **OKEQT** solutions to this problem, the introduction of an auxiliary unknown, as in Mateus C's (AH8) solution. The **y** he used to represent the amount of money left is not an essential *element* of the problem, once it can be totally avoided by the immediate use of the equality. Mateus's solution does not deal directly with this auxiliary unknown; rather, it plays a more *descriptive* role, although being clearly seen as a number (by belonging to the numerical-arithmetical context of the expressions). Whether he saw the two expressions on the left hand side of the two equalities as representing "calculations" or as true "complex" algebraic *objects*, one cannot infer from the script alone, but the notation certainly provides an environment where the latter is made easier.

$V = 5 \cdot x$ x = quanto restou

$S = 11 \cdot x$

$\begin{cases} 6900 - 5x = y \\ 12000 - 11x = y \end{cases}$

$6900 - 5x = 12000 - 11x$

$6x = 5100$

$x = 850$

$V = 5 \cdot 850 = 4250$

$S = 11 \cdot 850 = 9350$

Vitória 4750 e Sandra 9350

Eu determinei o preço de cada camiseta e quanto restou e com isto montei um sistema

Mateus C,AH8

On the other two solutions that employed an auxiliary unknown (again a **y**), the algebraic processing included its direct manipulation (Tathy G, AH8; Silvio S, AH8), once the two equations were primarily seen as a set of equations in two unknowns; Tathy says: "I did a *system of the 1st degree* [=*linear*]". Although not being the simplest solution — from the technical point-of-view — their approach shows exactly the ***internalism*** that is characteristic of *algebraic thinking*: the quantity represented by **y** was not required in the problem to be evaluated nor necessary to the continuation of the solution, and that those students were aware of that can be seen on the fact that they did not substitute the **x** back to determine **y**". Their solutions are quite characteristic examples of *thinking algebraically.*

V: 6900 → 5c

S: 12000 → 11c

$6900 - 5x = y$

$12000 - 11x = y$

$-5x - y = -6900$

$-11x - y = -12000$

$5x + \cancel{y} = 6900$

$-11x - \cancel{y} = -12000$

$-6x = -5100$

$x = 850$

Fiz sistema do 1º grau

Vitória gastou: 4250

Sandra gastou. 9350

Tathy G, AH8

$(-1)\ 6900 - 5x = y$
$12000 - 11x = y$
$(-1)\ -6900 + 5x = -y$
$12000 - 11x = y$
$5100 - 6x = 0$
$6x = 5100 \quad x = 850$

9350

Silvio S, AH8

One **WEQT** solution is of interest. Sergio P (AH7) writes down an equation that does not model the problem correctly, clearly for not understanding the problem's statement; he never bothered with the fact that **x** representing the price of a T-shirt, it would not be possible to begin with less money, to "add" less T-shirts and to end up with the same amount of money as the other person that had begun with more money and "added" more T-shirts. Then — and this makes the previous "disregard for the context" even more striking — he wrongly manipulates the equations (between the third and fourth lines) to produce a value for **x** that is positive, once *he knows it represents a price and thus has to be a positive number.*

$6900 + 5x = 12000 + 11x$
$+5x - 11x = 12000 - 6900$
$-6x = +5100$
$(-1)\ x = 5100 / 6 = 850$
$x = 850$

Sergio P, AH7

On the previous subsection (**Seesaw 4x problems**), we pointed out the importance of having a representation of the **T&E** *templates* in order to foster the process of transforming them into *objects*. Kelly L's (FM3) script shows, however, that there is a significant difference between the two types of representation, once the *equation* form might not convey the order of operations— as it indeed does not in the type of problem we are examining. Obviously, this problem can be overcome if the student has a good grasp of the process of evaluating numerical expressions.

Had / Bought
85 pounds - 11 Lp's — (11 x ? = (?)   (85 - ?) = price
67 pounds - 5 Lp's   11 x 1 = 11   85 - 11 = 74 } not the same
5 x ? = (?)   (67 - ?) = price   5 x 1 = 5   67 - 5 = 62
11 x 3 = 33   85 - 33 = 52 } the same
5 x 3 = 15   67 - 15 = 52
Came out with £52 after
Price of each LP is £3.00 each

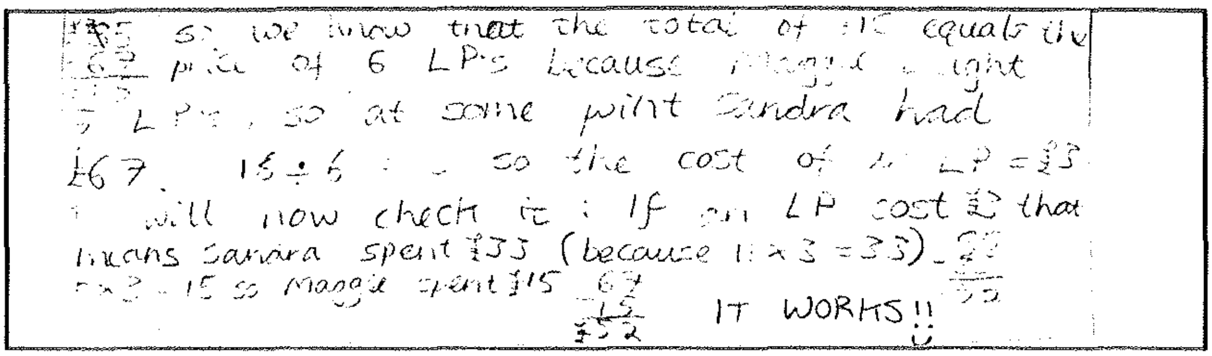
Kelly L, FM3

Of all **OKCALC** solutions to this problem, only one does not correspond to the scheme "the extra money Sandra had corresponds to the extra Lp's she bought, etc." (David W, FM3). Esther F (FM3) instead, reasoned in a manner similar to the "if George throws away 6 bricks and Sam does no throw away any..." described on the **Seesaw 11-5 problem** subsection. That only one solution employed such reasoning with **A11-5** problems, while a significant number of them appeared with **E11-5** problems, suggests that "objects" of the context of the problem become in fact *objects* in the model used to solve the problems, as the "balancing process" property is immediately associated with the Seesaw context but not with the Sale situation[21].

The answer is £3.00. First, I found the difference between the amounts that the girls took. Then, I divided that by 6 (the difference between the amount of LP's they bought). That gave me the answer.

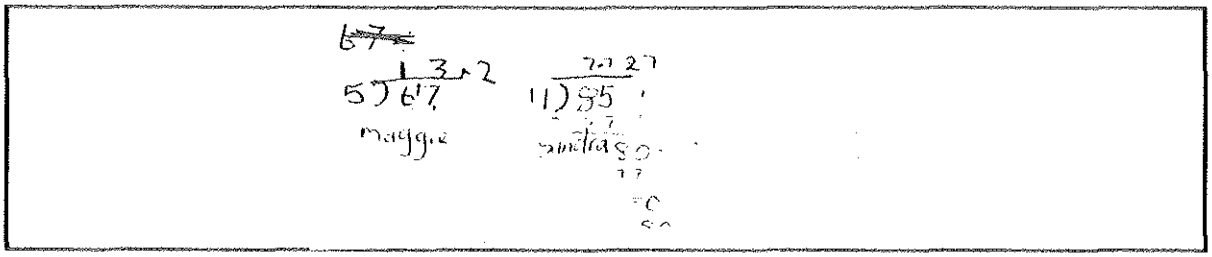
David W, FM3

[21]This "balancing process" property consists in the possibility of a gradual qualitative change in the balance state of the situation: the two sides of the seesaw being more or less near a balanced state or the difference between the money the two friends have being greater or smaller.



Esther F, FM3

Seven of the **WCALC** solutions take us in the same direction. In those solutions (eg, Shelley S, FM2) the students treat the problem as if both friends had spent all their money, and try to divide Sandra's money by the number of Lp's she bought and the same for Maggie to see if both divisions come to the same result. This type of solution did not appear on any **Seesaw 11-5** problems, most probably because it is quite obvious that the two friends will still be sitting on the seesaw when it is balanced, and this means that not all the weight will have been thrown away.



Shelley S, FM2

Of the remaining **WCALC** solutions, in four of them the **total money** is divided by the **total number of Lp's** — a strategy similar to dividing each friends' money by the number of Lp's she bought, but avoiding the possibility of having different priced Lp's for each friend — and the rest are attempts to produce a sensible answer from the numbers involved, some of them not very clear at all.

The Sale 4x problem

The most remarkable fact in relation to the solutions to this problem is that there is only one **OKCALC** solution (Keith W, FM3) out of a total of 82 students attempting it.

Keith's solution is unique in that he divided by **3** not because he modelled the problem with "1 lot, 4 lots" and concluded that "there is 3 lots more to Sandra", as one would expect, but instead he saw that Sandra would have to spend the difference between them (so they would be equal) and also some more money to allow for Maggie's expenditure; this means that the difference consists of three parts that will make four together with the extra part, that Maggie also gets.

85 − 67 = 18 Sandra spent £24 Maggie spent £6.
I found the difference and ÷ it by 3 because Sandra would have to spend that and £6 more (which is 1/3 of the difference) then Maggie would spend £6 which takes them to the same amount.

Keith W, FM3

This finding shows that it was very difficult, if not impossible for those students to establish the necessary *unit* that would allow them to use the "1part, 4 parts" strategy; the same situation was found with Seesaw **4x** problems, indicating the extent to which non-algebraic solutions depended on the existence of *parts and wholes which can be manipulated.*

The mistakes found on **WCALC** solutions to this problem represent mainly two aspects:

(i) not considering at all the relationship between what each of the two friends spent, thus focusing only on the difference between what they initially had (Joanna J, FM2),

67 − 85 = 18 Sandra spent £18
3×6 = 18 , Maggie spent £0
They bouth came out with £67

Joanna J, FM2

(ii) ignoring the fact that Maggie also spent one "lot" and dealing only with the 4 parts of Sandra (William C, AH7).

12000
− 6900
5100

5100 | 4
11 1275
30
20

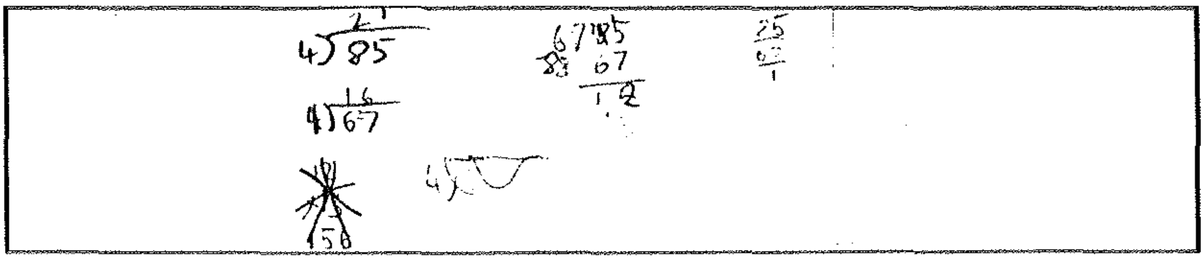
Sandra gastou 5100 a mais que VITORIA
Sandra 6,375
Vitoria 1,275

Tentativa.

William C, AH7

"Sandra spent 5100 more than Vitoria" and at the bottom line, "Attempt" (meaning probably that he was not sure of his solution)

As it had happened with **Sale 11-5** problems, there were a number of attempts to divide the **total money** by the **total number of parts** (Brian H, FM3), this being again a consequence of the possibility of the friends having spent all their money; only this time those attempts use only divisions by **4**, for the reasons explained above. In only two cases a division of one of the friends' money by **5** was used, in both cases taking the bigger initial amount (Sandra's). It might be that those students interpreted the "4 times as much" statement as meaning "4 parts more than" and this produced the need to consider one extra part.



Brian H, FM3

One of the **OKEQT** scripts (Fabiana M,AH7) provides an important insight on how the ability to solve "algebraic word problems" in general can benefit from the ability to think algebraically, and we do not mean, of course, the possibility of developing "automatic" solution procedures. In Fabiana's script it is immediately clear that she thought first of all of the existence of an unknown quantity — most probably a habit developed through the use of equations; we have already seen that in a problem like the [4x] problems this comes to be an essential step to reach a correct solution. Although the availability of a special notation certainly promotes a better grasp of that notion (Fabiana: "...I thought of an unknown (x)..."), we must keep in mind that it is the *analytical* character of the algebraic method that produces the need to make the unknown into an *object*.

V= 6900
S= 12.000.

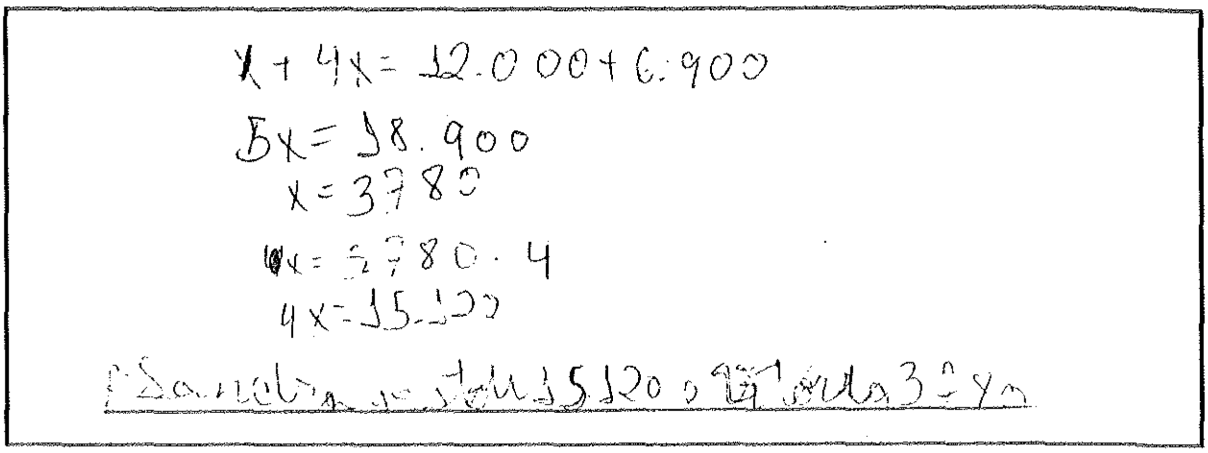
$$6\,900 - x = 12.000 - 4x.$$
$$-x + 4x = 12\,000 - 6\,900$$
$$3x = 5.100$$
$$x = \frac{5.100}{3} = 1700$$

$$\begin{array}{r} 6900 \\ -\;1700 \\ \hline 5200 \end{array} \qquad \begin{array}{r} 12.000 \\ -\;6800 \\ \hline 05200 \end{array} \qquad \begin{array}{r} 1700 \\ \times\;4 \\ \hline 6800 \end{array}$$

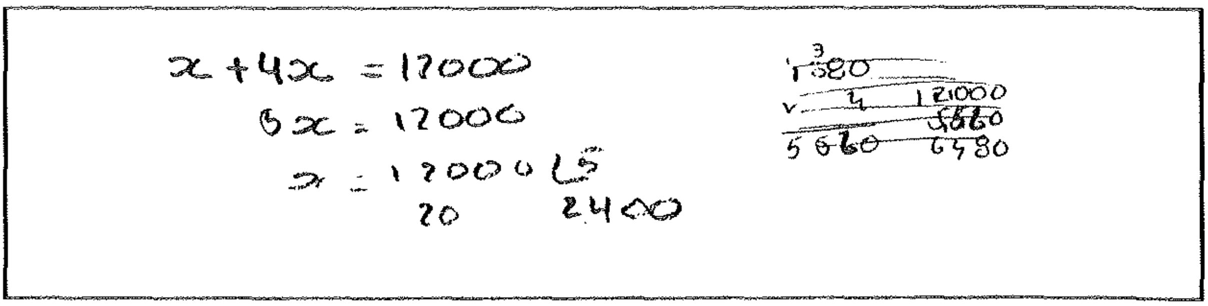
O problema quer saber quanto V e S gastaram, logo pensei numa incógnita (x). O problema ainda dá um dado: S gastou 4x mais que V. Logo lembrei da sentença que aprendi em geometria e álgebra. Aí ficou fácil.

Fabiana M, AH7: "The problem wants to know how much V and S spent, thus I thought of an unknown (x). The problem also gives an information: S spent 4 x more than V. Then I remembered the sentence that I learned in geometry and algebra. It then became easy."

Four of the **WEQT** solutions reproduce in the wrong setting of the equations, some of the mistakes we observed with **WCALC** solutions. Fernando C (AH8), for example, equalises the **total number of parts** to the **total money**, and correctly solves the equation and Sidnei A (AH7) attributes **5** parts to Sandra (the "1 and 4" mistake we discussed 3 paragraphs above).



Fernando C, AH8

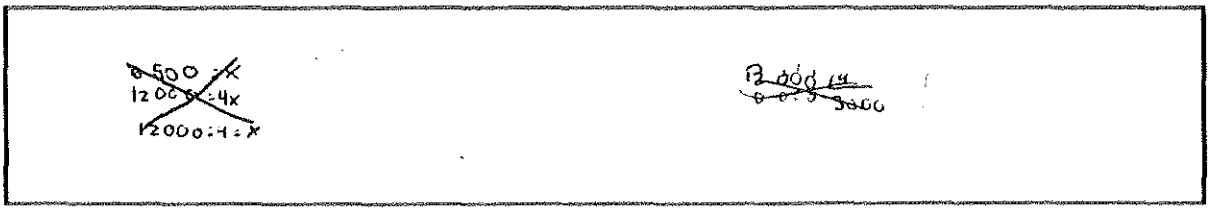


Sidnei A, AH7

One has to be amazed by Luis N's (AH7) attempt, as he writes <u>on the first line</u>

$$6500 = x$$

without immediately concluding that the solution to had been found. We think that he had in fact structured the problem by attributing one part to Vitoria's total money and 4 parts to Sandra's total money, as some students did with the **Sale 11-5** problem, and that the algebraic notation was not being seen by him—at that point—as representing true equations to be solved. He then seems to move away from this initial interpretation and "solves" the second equation, and that is when he realizes that the two values for **x** do not agree, and something must be wrong.



Luis N, AH7

This *shift of interpretation*, so dramatically illustrated by this script, is certainly at the core of using algebra to solve contextualised problems: the equation is set by transforming series of calculations — *analogically* associated with the problem's "story" or context — into arithmetical expressions[22], and then those expressions are linked by equalities —again, *analogically* associated with the context. It is only then that it is treated *internally*, as an equation, and this shift, by marking the transition to a different *Semantical Field* marks also the passage to a distinct *mode of thinking*.

SUMMARY OF FINDINGS AND CONCLUSION

An aspect of the non-algebraic models used by the students emerged clearly from the analysis of this group of scripts: their *synthetical* nature, with the process of solution always proceeding from the known values to the required unknown one through a series of *evaluations*. The few exceptions would be those solutions to **E11-5** where there is a hypothetical manipulation of the situation that leads to the "only 6 bricks need to be removed from George's side and none from Sam's side" structure.

Another conclusion to be drawn from the analysis of this group of answers is that many students did not see numerical-arithmetical expressions and equalities as *objects* that could be manipulated on themselves to produce further useful information in the process of solving the problem. This aspect was particularly crucial in relation to the **SN1** problem, that is, as we saw, very difficult to be modelled into a *geometrical* or *comparison of wholes* model, and thus the inability to see numerical-arithmetical expressions as informative led to very low facility levels among the English students. That those same students did significantly better on the contextualised problems, shows that *the non-algebraic methods used by them is based to a great extent in the perception of* ***parts*** *which can be manipulated, and that the choice of arithmetical operations to be performed is almost completely dependent on the manipulation of non-numerical objects*; the numbers in the

[22]**At this stage** those expressions are in fact *arithmetical*, once the unknown numbers are treated as if they were known, as we have already seen, and they are seen as calculations to be carried out.

problems were rather seen as *measures*. The greater difficulty with [4x] problems, in comparison with [11-5] problems also provides a clear support to this conclusion. To put it in terms of our framework, ***those students that failed to solve the SN1 problem but could handle the contextualised problems were unable to operate within the Semantical Field of numbers and arithmetical operations***. Moreover, it was difficult for many students — probably most of those not using an algebraic approach — to move away from the *Semantical Field* where the problems were originally set, eg, to model a contextualised problem with a *comparison of wholes* model. They kept strongly attached to the original "icons" provided with the problems' statements and consequently limited their perception of the problems' structures to what is more ordinarily associated with those contexts.

Moreover, the non-algebraic solutions, correct or not, were characterised by their *contextwise homogeneity* in relation to addition and subtraction of measures. This is an important aspect for two reasons. First, because it points out to a possible important source of information used by those students on what can or has to be done to solve a given problem. Second, because if this is indeed a deeply rooted informative pointer in a person's problem solving schemes, it would certainly be difficult to operate on a *Numerical Semantical Field*, where such pointers are truly meaningless. As a consequence, it might be that teaching "intuitive", "contextualised" or "localised" strategies for solving algebra word problems builds in fact a huge obstacle to be overcome when the "algebra time" arrives, and this suggests that an early start with the algebraic approach might be of great help to reduce the difficulties with the learning of algebra, not because of the "extra time to practice", but because of the earlier development of a degree of independence from such pointers[23].

Still in relation to the influence of schooling in the development of an algebraic mode of thinking, we found it very significant that the "default" approach for Brazilian 7th graders was non-algebraic — although they were able to use an algebraic one — while for the 8th graders the "default" approach was an algebraic one ; that the same was not found in relation to the corresponding English groups, and that a considerable similarity of ages existed, strongly suggests that the development of *algebraic thinking* is a process

[23]Obviously, those pointers are not useless in all situations, and they may even be of great help when one is trying to make sense of the relationships involved in a more complex task or problem. What we imply here, is that **both** "homogeneity bound" and "not-homogeneity bound" strategies should be made available and equally developed. Once much of everyday activity is indeed "homogeneity bound", we suggest that schooling could avoid the development of a too strong primacy — eventually a pernicious one — by offering an early alternative way of thinking.

much more akin to ***cultural processes*** than to age-related stages of intellectual development.

The analysis of the scripts for this group of questions threw much light on different uses of algebraic notation and on possible consequences of resorting to the notion that setting up an equation to model a problem is a *translation* process. Students used letters both in a truly algebraic way — to denote numbers — and in syncopated forms of the verbal statement. The latter use caused two types of difficulty:

(i) as letters were used as an abbreviation of the verbal text, and there was a context to support this usage, different quantities — different at least in principle — ended up being represented by the same letter; also, this usage sometimes introduced new "unknowns" (as, for example the individual weights of each friend on the seesaw);

(ii) as one "describes" a sequence of things happening, no care has to be taken to match the order of the verbal syncopation with the conventions of numerical-arithmetical expressions — which are not necessarily useful if one is simply trying to make the statement more comprehensible by breaking and syncopating it, and both conventions are very distinct in most cases. Also, the objects involved are not numbers, but objects of the context (as we said, numbers are seen as *measures* and *operators*), and one should reasonably expect the subject to manipulate the letters — in fact *icons* of those objects — according to the properties he or she sees as relating to the objects those *icons* refer to; there is no *shift of referential*, no passage to another *Semantical Field.*

It seems, on the other hand, that the use of **standard** algebraic notation—instead of more iconic forms like boxes and question marks—might be of use to promote a more immediate ***transformation*** of a contextualised problem into an algebraic one, for example through the association between "**x**" and "**the unknown**", one immediate advantage being, as we saw with the [4x] problems, to make easier to overcome the difficulty of having to establish *units* that do not correspond to objects of the context.

Another important aspect to emerge from the algebraic solutions offered, is that we could distinguish levels of *sophistication* in the processing of the algebraic models used to model the problems. The introduction of auxiliary unknowns, the use or not of "standard forms" of equations in the process of solution, a more or less restricted use of negative numbers, "one step-one line" solutions and more flexible ones, and above all, some solutions that treated the equation as a whole (eg, multiplying a whole equation by -1)[24], instead of the more limited perception of thinking only in terms of "chunks" (eg, breaking the equation down into 273, -11x, =, 181, -5x, and seeing those as the blocks to be dealt with). In all cases, however, the same basic characteristics that our theoretical characterisation of algebraic thinking established can be identified: *internalism*, *arithmeticism*, and *analiticity*.

4.4 CARPENTER-CHOCOLATE-SETS OF EQUATIONS PROBLEMS

THE PROBLEMS

I am thinking of two secret numbers.
I will only tell you that...

(first no.) + (second no.) = 185
and
(first no.) - (second no.) = 47

Now, which are the secret numbers?
(Explain how you solved the problem out and why you did it that way)

Sets 1-1

I am thinking of two secret numbers.
I will only tell you that...

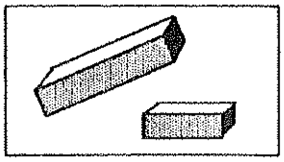
(first no.) + (3 x second no.) = 185
and
(first no.) - (3 x second no.) = 47

Now, which are the secret numbers?
(Explain how you solved the problem and why you did it that way)

Sets 1-3

[24]Of course this corresponds ***formally*** to multiplying each side of the equation by -1, but we are dealing here with the perception of algebraic objects and their properties, and not with a strict formal justification.

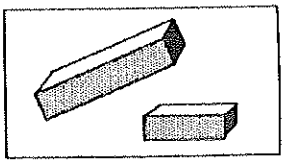
At the right you have a sketch of wooden blocks.
A long block and a short block measure 162 cm altogether.
A short blocks measures 28 cm less than a long block.



What is the lenght of each individual block?
(Explain how you solved the problem and why you did it that way)

Carp 1-1

At the right you have a sketch of wooden blocks.
A long block put together with two of the short blocks measure 162 cm altogether.
If two short blocks are put together, they still measure 28 cm less than a long block.



What is the lenght of each individual block?
(Explain how you solved the problem and why you did it that way)

Carp1-2

At Celia's shop you can buy boxes of chocolate bars or you can buy spare bars as well.

A box and three spare bars cost £8.85.
A box with three bars missing cost £5.31

What is the price of a box of chocolate bars in Celia's shop? What is the price of a single bar?
(Explain how you solved the problem and why you dit it that way)

Choc

GENERAL DESCRIPTION

This group of problems was developed with the objective of:

(i) examining students' strategies to solve "secret number" problems involving two secret numbers and to compare those strategies with the ones used with the corresponding contextualised problems; each of the secret number problems in this group corresponds to one or two contextualised problems and the relationship between the models employed on a secret number problem and its correspondent contextualised problem(s) will be closely examined. Both secret number problems were set in a *normal form* of sets of simultaneous equations, given in a syncopated, rather than literal, notation; the use of symbols for arithmetical operations and for equality — as opposed to the traditional verbal formulation[25] — was intended to keep the problem as close as possible to the *Numerical Semantic Field* and to allow us to examine to what extent those *numerical-arithmetical* statements made sense to the students.

(ii) examining the effects of an increase in the structural complexity of a problem in the strategies used;

As we will show, it was easier with this group of problems than with the previous ones to distinguish *algebraic* and *non-algebraic thinking* even in the context of a solution using algebraic symbolism to describe and control a non-algebraic process, once the students were more generous with the explanations provided with their answers , and those explanations were in general of a much better quality, this being particularly true for the contextualised problems.

DISCUSSION OF POSSIBLE SOLUTIONS

Chocolate Box problem (Choc)

This problem seems to inevitably involve two unknowns.

An algebraic model is

$$\begin{cases} x + 3y = 8.85 \\ x - 3y = 5.51 \end{cases}$$

where **x** is the price of a box of chocolate bars and **y** is the price of a single bar. The most likely solution to this set of equations is to add the two equations to produce

$$2x = 14.36$$

and to solve it from there.

[25]Eg, "I am thinking of two numbers. If I add the two of them the result is ...," and so on.

Two non-algebraic models seem possible here:

(i) *"The first box has 6 bars more than the second, so, if I work out the difference between the two values* [8.85 and 5.51] *I will have the price of 6 bars"*, etc.

(ii) *"If I put together the two boxes* [the one with extra bars and the one with bars missing] *the three extra bars on the first box can be transferred to the second box, making two complete boxes. So, if I add the two prices I will have the price of two boxes"*, etc.

It is central that with the non-algebraic models, the choice of operations to perform is totally subordinated to the manipulation of the image of the boxes and the bars. Also, on those models one thinks of **two boxes and three bars** and not of **the price of a box and the price of a bar used in different places**. Moreover, the divisions that would follow (by 6 or by 2, respectively) would certainly be a way of *evaluating* the sharing of an amount of money into the corresponding number of parts.

Another possible *analogical reasoning* would be,

(iii) *"If one box with 3 bars missing cost 5.51, then a box costs 5.51 plus 3 bars"* and proceed to *"then, 5.51 plus 3 bars with the extra 3 bars cost 8.85"*, etc.. This reasoning could both produce a direct solution, through the manipulation of the *whole-part* relationship, or lead to the single equation

$$(5.51 + 3y) + 3y = 8.85$$

This approach is substantially different from both (i) and (ii), as the meaning of the "plus" in "5.51 *plus* 3 bars" can only be understood in the context of *prices* ("3 bars" ≈ "the price of three bars", while in (i) and (ii) "bars" stand for bars, as we saw. If one writes

$$\textbf{1 box - 3 bars = 5.51}$$

the "=" sign reads "cost" and means that the object on the left is *labelled with the price 5.51*. On the other hand, if one writes

$$\textbf{1 box = 5.51 + 3 bars}$$

the equality has to be interpreted as meaning an equality between prices, if not pure numbers. Reading the "=" sign as "costs" produces a somewhat puzzling phrase, very similar to the one in the well-known riddle "a fish's weight is 10 pounds plus half a fish...".

If the *shift* in the interpretation of the equal sign in the two *written* sentences can be made bearable by the ambiguous use of the equal sign, it corresponds in fact to a change in the type of relationship that is being considered, and it seems to offer a substantial obstacle to be overcome within the *Semantical Field* of chocolate boxes and bars in which the problem is set, and one has to remember that it is within this *Semantical Field* that the manipulation producing "1 box = 5.51 + 3 bars" from "a box with 3 bars missing costs 5.51" would have to happen, ie, the manipulation would have to occur *before* the sentence being written.

The substitution of the resulting sentence into the first line of the problem's statement, to produce "(5.51 + 3 bars) + 3 bars = 8.85" would also be problematic, as the substitution of the "actual" box by its price would require a strong shift in the understanding of the original statement (with the added difficulty that the price replacing the object is stated in terms of another object's price).

The importance of analysing possibility (iii) in some detail is that within the *Semantic Field of numbers and arithmetical operations* the manipulation

$$\begin{cases} x + 3y = 8.85 \\ x - 3y = 5.51 \end{cases}$$

$$x - 3y = 5.51 \Rightarrow x = 5.51 + 3y$$

$$\therefore \quad (5.51 + 3y) + 3y = 8.85$$

presents none of the difficulties discussed above, which is a clear indication that (a) within the *Semantic Field* of the chocolate boxes and bars the *objects* one deals with are completely distinct from those one deals with within the *Semantic Field of numbers and arithmetical operations* — and thus the types of relationship involved and the requirements on a notational system — and (b) *arithmetical internalism*, a most central characteristic of thinking algebraically, allows one to operate continuously without having to consider shifts such as those we have just discussed. We have here a very fine example of the fact that a compact notation is possible if one is thinking algebraically, exactly because of the *homogeneity* produced by the *arithmetical internalism.*

Solutions (i) and (ii) above, resemble very much the strategy of adding or subtracting the two equations in a set of equations. Nevertheless there is a fundamental difference between the two processes. In solution (i) the full boxes are thoroughly ignored, and the conclusion that the first box has six bars more than the second box comes from a "counting up"[26] strategy, rather than from "subtracting" the second line from the first, once it is obvious that the "taking away" meaning of the subtraction would make no sense in this situation because of the need to "take away what is already missing". In solution (ii), what is done in fact is a *transfer* of the three extra bars in the first box to fill up the second box;

[26]Evaluated, of course, with an addition. The full box works, in fact, as a form of "zero level."

the extra bars in the first box are never operated with the missing bars in the second box. Finally, in the additive solution of the set of equations **-3y** is ***numerically*** added to **3y** and *the terms cancel each other out because the result is zero.* Similarly for subtracting the second line from the first. The point to be made here is that although solution (ii) "written" using algebraic notation is actually indistinguishable from a true algebraic additive solution of a set of equations modelling the problem, the two solutions are essentially distinct, each one being the result of operating within a different *Semantic Field.*

Carpenter 1-1 problem **(Carp1-1)**

Two algebraic models seem more likely to be used to model this problem. One is the set of equations (L stands for the length of the longer block, S for the length of the shorter block)

$$\begin{cases} \mathbf{L + S = 162} \\ \mathbf{L = S + 28} \end{cases}$$

and the other is the single equation

$$\mathbf{(S + 28) + S = 162}$$

It is obvious that by a substitution, one will arrive from the set of equations at the same single equation, but by separating the two models we want to emphasise that the substitution can be made within the *Semantic Field of numbers and arithmetical operations* (from the set of equations to the single equation) or within the *Semantic Field of the Wooden Blocks* (the longer block being *represented* as a short block with an extra bit added to it). It is clear that in the latter case the "+" sign means "conjoining" and not the arithmetical operation.

From the results obtained on the exploratory study we expected non-algebraic solutions to this problem to be of one of two types[27] (figure CCS 1, for (i), a similar diagram for (ii)):

(i) "*if I cut 28 out of the longer block I will have 2 equal [short] blocks, so if I take 28 from the total, I will be left with the length of two short blocks...,*" etc.

(ii) "*I cut the total in two, take away 14 from one half and add it to the other half, thus making the difference 28.*"

[27]The original problem in the Exploratory Investigation had a slightly different form from this one, but we still expected the solutions to follow the same pattern.

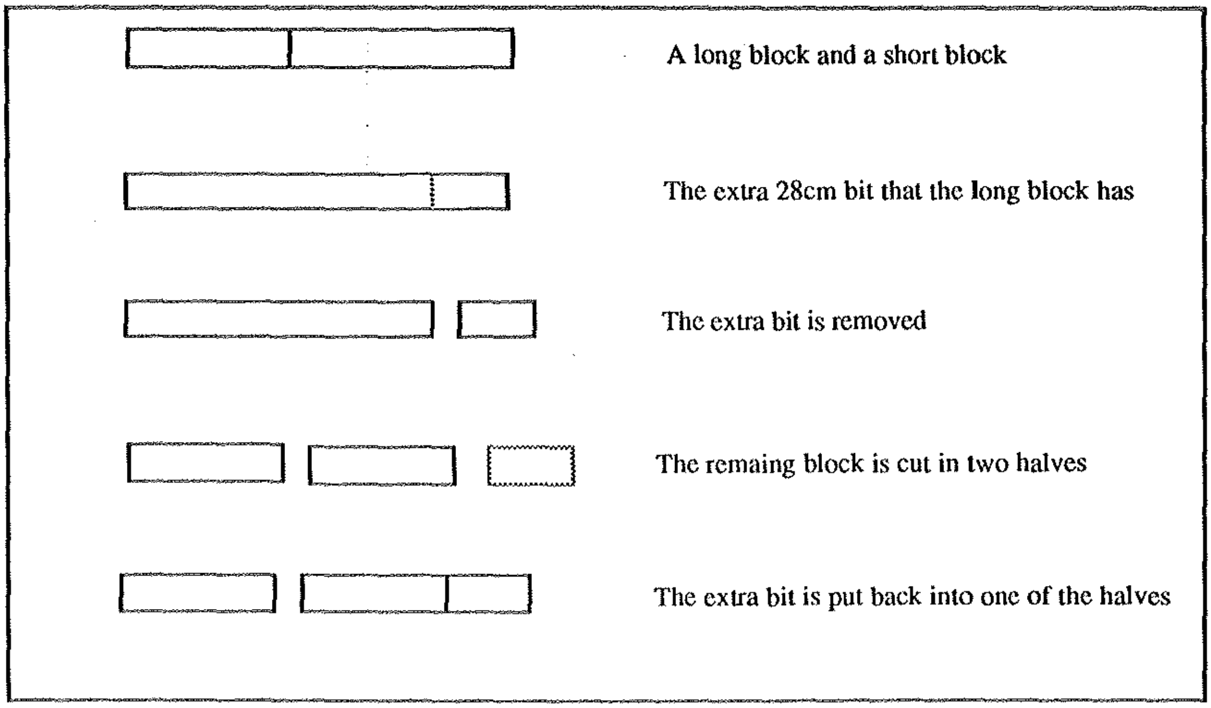


fig. CCS 1

Again, in those non-algebraic solutions the choice of operations to be used would be totally guided by the manipulation of the objects of the context, eg, a subtraction to *evaluate* how much is left after a bit 28cm long is cut from the total.

From a script containing only equation(s) without any other explanation, it would be virtually impossible to distinguish solution (i) above from an algebraic solution using a single equation.

Carpenter 1-2 problem (**Carp1-2**)

As for the Carpenter 1-1 problem, the two likely algebraic models would be a set of equations

$$\begin{cases} L + 2S = 162 \\ L = 2S + 28 \end{cases}$$

or a single equation

$$(2S + 28) + 2S = 162$$

Also, the same non-algebraic procedures could be used, with the additional step of "slicing" the shorter block in **Carp1-1** into the two required smaller blocks. The additional difficulty that appears in **Carp1-2** is that non-algebraic solutions similar to those presented a few paragraphs above for **Carp 1-1** would have to deal with the "complex" object "two short bars" replacing the "short bar" in **Carp1-1**.

Secret Number problems (**Sets1-1** and **Sets1-3**)

Those two problems could be represented by the sets of equations

$$\begin{cases} x + y = 185 \\ x - y = 47 \end{cases}$$

and

$$\begin{cases} x + 3y = 185 \\ x - 3y = 47 \end{cases}$$

presented in a more "syncopated" form.

The standard algebraic solutions would be:

(i) adding the two equations and solving the resulting equation for **x**, etc., and

(ii) isolating one of the variables from one of the equations and substituting in the other, etc..

As with the **SN1** problem in SSE, non-algebraic solutions to those problems would involve modelling the problem's statement into a non-numerical *Semantic Field*, for example for **Sets1-1**:

> "Altogether they are 185, and the second number is 47 less than the first one.
> So, if I take 47 from the 185 it is like having two of the second numbers...,"
> etc.

which of course corresponds to a structure similar to the one depicted on figure CCS 1. The specific model described above involves the additional difficulty of interpreting

(first secret no) - (second secret no) = 47

as meaning

(first secret no) = (second secret no) + 47

Seen within the *Semantical Field of numbers and arithmetical operations*, it is a simple equivalence, but when seen as a transformation of *whole-part* relationships — where the subtraction means "removal" and the addition means "conjoining" — the equivalence is not as direct as before, because each expression involve a subtle but significantly different representation; the main difference would be that on the first expression the difference is the *result* (or *final state*) *of an action*, while on the second expression it is either the *initial state* or *first operand*, or the *operator parameter* or *second operand*, depending on which model is used. As we will see in the analysis of the problems in the **Buckets** group of problems, students can easily produce the transformation

$$\mathbf{x + a = b \quad => \quad x = b - a}$$

in the context of a secret number problem <u>if a and b are known and b>a</u>, which suggests that this difficulty is strongly linked to the fact that the required transformation does not produce or permit any *evaluation*.

GENERAL DATA ANALYSIS

The performance of the Brazilian group AH7 is much superior than that of the age-corresponding English group, FM2, and in fact it is comparable to that of the older FM3 group. In relation to the last group of problems, we saw that FM2 performed better than AH7 on the contextualised [11-5] problems, where the context objects were more readily available and performed worse on [4x] problems, where the meaningfulness of an arithmetical relationship (derived from the 1 to 4 ratio) was shown to be a crucial factor in successfully solving those problems. Here this should not be a relevant factor, because all the parts and relationships in the three contextualised problems are explicitly given and only *conjoining*, *taking away* and *sharing* are sufficient to model these problems non-algebraically.

Another interesting aspect of AH7 students' performance is that their approach is clearly non-algebraic on the contextualised problems (which can be seen on both correct and incorrect answers), but on the **Sets** problems the preferential approach shifts to an algebraic one, a feature more clearly seen on the choice of strategies used in incorrect solutions (for the contextualised problems, all the incorrect solutions are **WCALC**; for **Sys1-1** the incorrect solutions are almost equally divided between **WCALC** and **WEQT**, and for **Sys1-3** most of them are **WEQT**). This behaviour corresponds well to a similar behaviour observed on the SSE group problems, and it suggests that those AH7 students had a more selective approach to the choice of strategies than the students on the AH8 group.

That almost no **OKCALC** solution for the sets of equations appeared, offers further support to our conclusion that it was extremely hard for those students to *model-back* the numerical-arithmetical statements into a non-numerical *Semantical Field,* as we had observed with the Secret Number problem on the Seesaw-Sale group. Although the complexity of the problems' statement is certainly an issue here, we think that it is not a crucial one, once the facility level for the contextualised problems is significantly higher than on the **Sets** problems on AH7 and on FM3, showing that they could to some extent cope with the complexity offered by those problems. We think that two factors have to be taken into consideration. First, the difficulty in extracting information from the numerical-arithmetical relationships on what can and should be done to solve those problems, ie, the *lack of meaning* of those expressions, which would indicate that those students could not operate on a *Semantical Field* where those expressions were numerically meaningful by themselves. Second, the fact that "the first number" was greater than the "second number" or "three times the second number" was expressed by a subtraction, and our results suggest that a non-numerical interpretation of such a subtraction is much harder than a non-numerical interpretation of addition in the context of comparing measures.

Two points arise the from analysis of the use of equations and sets of simultaneous equations by students on AH8 to solve the contextualised problems[28]:

(i) on **Choc** all **OKEQT** solutions (47%) used sets of equations. The form in which **Choc** was introduced, with two "conditions" or "statements" clearly distinguishable, two unknowns clearly distinguishable, and a visual presentation strongly resembling sets of equations (eg, the two conditions written on bellow the other) strongly suggested the "sets of equations" approach, at the same time it discouraged the direct modelling into one single equation; in fact 12% of those **OKEQT** solutions to **Choc** proceeded from the set of equations by a substitution, but this procedure was never used *before* the statement had been represented in algebraic notation. This shows that what was not seen as *meaningful* in the *Semantical Field* of the chocolate boxes became visible in the *Numerical Semantical Field* (as we had indicated in the analysis of possible models).

(ii) the greater complexity of the conditions in **Carp1-2** made a *direct non-algebraic substitution* leading to a model with a single equation much more difficult; as a result, the separate representation of the two relationships usually preceded their manipulation. This is absolutely clear from the fact that one has, for **Carp1-1**, 47% of

[28]We restrict our analysis here to AH8 because this was the only group to consistently use this approach.

solutions from a single equation and 32% of solutions from a set of simultaneous equations, but for **Carp1-2** the percentages change to only 5% of single equation solutions and 42% of sets of simultaneous equations solutions

A possibly relevant mistake was made when producing the Brazilian version of **Carp1-2**, as the original phrase "If two short blocks are put together, they still measure 28cm ***less*** than a long block" ended up as the equivalent of "The long block is 28cm ***longer*** than two short blocks put together." In **Carp1-1** both Brazilian and English versions used the former form. Nevertheless, this difference in the statement did not seem to produce significant effects on the results, as in **Carp1-2** AH7 kept at a substantially higher level than FM2, and AH8 kept at a higher level than FM3 — as it happens for both pairs of corresponding groups in **Carp1-1**.

The biggest fall in the facility level from **Carp1-1** to **Carp1-2** is for AH8 (from 90% to 52%), and it is associated with a much greater difficulty in producing a single equation by a *direct non-algebraic substitution*; this failure to directly reduce the problem was not compensated by an increase in the proportion of non-algebraic solutions, but only by a moderate increase in the number of solutions using a set of equations. This shows again the lack of flexibility on the problem-solving behaviour of AH8[29]. In AH7, the fall in the facility level is smaller but still significant (from 69% to 44%), and it corresponds mainly to a smaller proportion of **OKCALC** solutions. In FM3 the facility levels are more similar (64% to 52%), and in FM2 practically nil (6% in both cases, for a sample of 17 students, ie, one correct solution for each of the two problems).

STUDENTS' SOLUTIONS

The Sets1-1 problem

All but two **OKEQT** solutions to this problem were produced by solving the set of equations directly suggested by the problem's statement. One of those two solutions employing a single equation, however, provides a good example of a *direct non-algebraic substitution*, with the added relevance of the descriptive use of literal notation (Mairê M, AH8).

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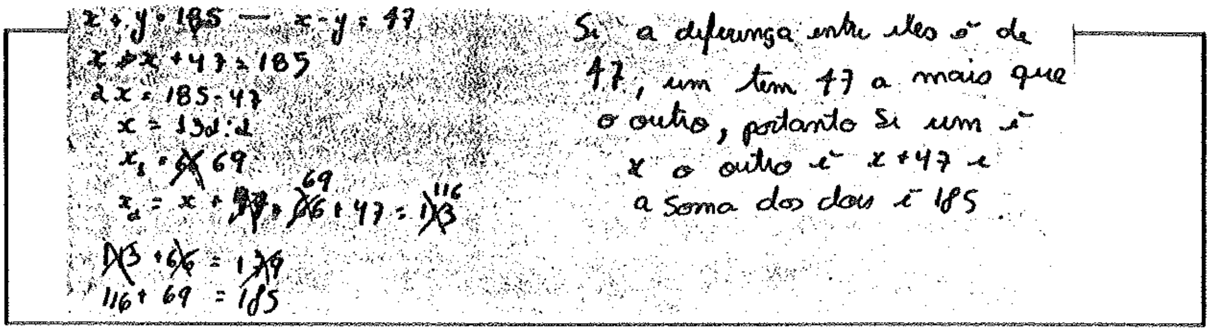
[29]Also, the proportion of **WCALC** solutions remains the same and that of **WEQT** increases dramatically.


x+y=185 — x-y=47
x+x+47=185
2x=185-47
x=138:2
x_1=69
x_2=x+47=69+47=116
116+69=185

Si a diferença entre eles é de 47, um tem 47 a mais que o outro, portanto Si um é x o outro é x+47 e a Soma dos dois é 185

Mairê M, AH8

"If the difference between them is of 47, one has 47 more than the other, thus one is x and the other is x+47 and their sum is 185."

Normally, from the script alone it would not be possible to decide whether the *direct substitution* was *non-algebraic* or *algebraic*, ie, whether it was respectively based on *modelling back* the second expression into, for example, a two sticks situation, one longer than the other, or a non-written manipulation of the second "equation". At first sight it seems the second is the case, as Mairê wrote down the two equations first (top left) and solved the problem algebraically before writing down the explanation (which is to the right of the algebraic solution). One detail of the solution, however, clearly suggests that she was not dealing directly with the equations she had written: her second equation (first line, after the m-dash) says that "the difference between the two numbers is 47" but it also implies that "x is the greater of the two". Nevertheless, on the second line she writes

$$x + x + 47 = 185$$

and not

$$y + 47 + y = 185$$

as it should be the case were she actually dealing with the equations written on the first line as objects being manipulated[30]. Although it is truly possible that the property she evoked to substantiate the substitution was seen by her purely as a property of numbers, we are led to the conclusion that in fact she was using a non-algebraic model, as it took her away enough from the equations' context to allow a complete shift in the meaning of the symbols used.

Andrea M's (AH8) solution, on the other hand, clearly exemplifies the *algebraic* substitution, done within the context of the algebraic model, ie, *after she had produced the algebraic model, and the substitution being meaningful within that* ***Semantical Field.***

[30]In this case it is obvious that this procedure did not affect the correctness of the solution, once in fact the actual algebraic solution begins at the second line, and not at the first, as it would seem to begin.

$$\begin{cases} n+s=185 \\ n-s=47 \Rightarrow n=47+s \end{cases}$$
$47+s+s=185$
$2s=185-47$
$2s=138$
$s=69$
$n=47+69$
$n=116$

é o mesmo processo da questão 3, só que o enunciado, agora, está sob forma de sistema.
Antes de separar variável de número é preciso deixar uma só variável, e esse processo é feito por substituição. depois é só separar um do outro.

Andrea M, AH8

"it's the same process as in question 3 [31], but only this time the statement is on the form of a *system*[32].

Before separating the variables one has to leave only one variable, and this process is done by *substitution* then it is only separating one from the other." (our emphasis)

Eurico G's (AH8) solution shows another procedure to reduce the set of equations into a single equation with one unknown, using "...the criteria of *comparison.*" [33]

$$\begin{cases} x+y=185 \\ x-y=47 \end{cases}$$
$x=185-y$
$x=47+y$
$185-y=47+y$
$2x=185-47$
$x=69$

$y=x-47$
$y=69-47=22$
R: Eles são 22 e 69

Eu resolvi, usando um sistema, pegando o que foi dado no enunciado e substituindo os nº secretos por incógnitas. Usei o sistema no critério de comparação.

Eurico G, AH8

Moreover, it shows that he directly attached an *arithmetical* meaning to the "+", "-" and "=" signs, as it is indicated by him saying that "I solved using a system, *taking what was given in the statement and substituting the secret numbers by unknowns*" (our emphasis). On his solution one can also see the importance of *internalism* in thinking algebraically, once the production of the expressions

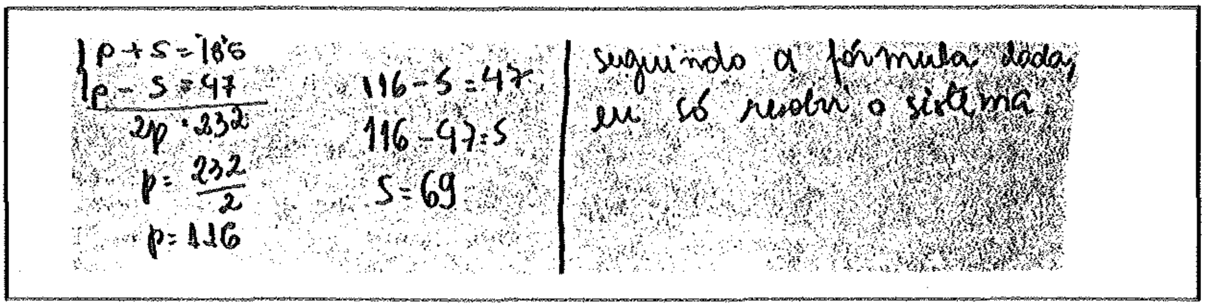
$$x = 185 - y \quad \text{and} \quad x = 47 + y$$

is meaningful only in the context of the *method of solution.*

---

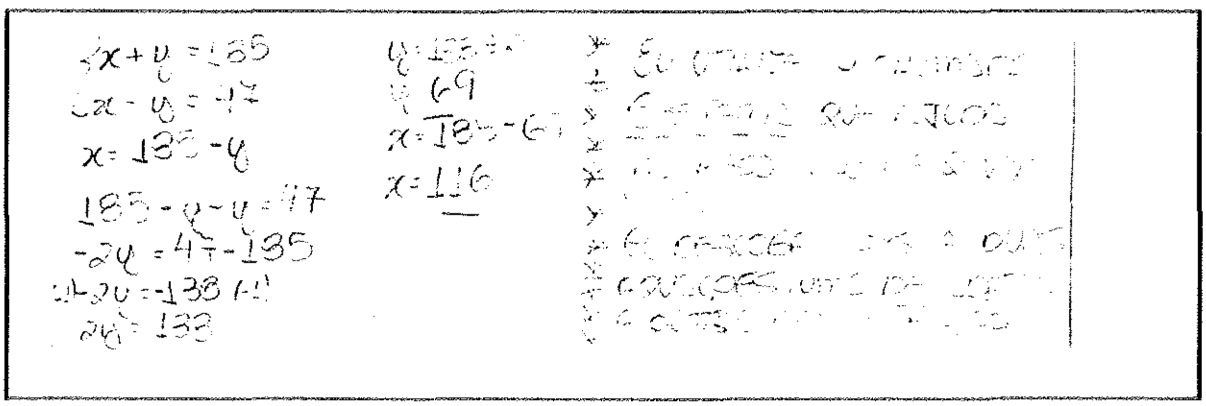
[31] We believe that she mistakenly referred to question 3 (**SN1**), having in fact intended to refer to question 2 (**Carp1-2**), which she solved using a set of equations.
[32] In Portuguese, *system of equations* stands for *set of equations*.
[33] *Comparison* being the "official" name for that strategy according to Brazilian textbooks.

Eurico's was the only **OKEQT** solution to use the *comparison* strategy. All the others used either *addition of equations* (eg, Erika M, AH8) or *substitution* (Andrea M, AH8, script already shown) strategies, with twice as many *substitution* solutions as *addition of equation* ones. *Formally*, the *addition of equations* strategy involves a more sophisticate algebraic perception than the *substitution* strategy, as one would have to perceive the equations as an *objects* that can be operated with. Nevertheless, one can actually perform the addition of the two equations *term by term*, with the correctness of the procedure being guaranteed by a trust in its algorithmic side rather than a deeper understanding of the procedure's roots.



Erika M, AH8

The solutions by Bruno N (AH8) and Alberto SA (AH8) also throw light into how students might identify the adequacy of using an algebraic strategy — in this case solving a set of equations. In Bruno's case it is the structural aspect that provides the hint (identifying equations, operations involved and variables), and in Alberto's case it is the direct recognition of equations in the problem's statement (as in Eurico's case, analysed above) together with the visual aspect ("...2 equations one bellow the other.").



Bruno N, AH8

$$\begin{cases} x+y=185 \\ x-y=47 \end{cases}$$

$x=185-y \rightarrow 185-69=116$

$185-y-y=47$

$-2y=47-185$

$-2y=-138$

$y=69$

1º número: 116

2º número: 69

Alberto SA, AH8

From the six **WEQT** solutions, three are of greater interest.

Ricardo G's (AH8) makes an almost careless mistake by "forgetting" to include the second **y** when he substitutes into the first equation the expression for **x** obtained from the second equation. Apart from that his solution is neat and correct, and had he checked his answer, he would have probably spotted the mistake and corrected it.

$$\begin{cases} x+y=185 \\ x-y=47 \Rightarrow x=47+y \end{cases}$$

$47+y=185$

$y=185-47$

$y=138$

$x+138=185$

$x=185-138$

$x=47$

Ricardo G, AH8

In Nicola D's (FM3) solution, the derivation of the three expressions

$$\mathbf{A = 185 - B \ , \ \ B = 185 - A \ \ \text{and} \ \ B = A + 47}$$

is technically correct, but she never gets any further. In a sense it seems that she was trying to put the expressions in a form in which she could see how to proceed, being unaware that from any of the expressions involving two unknowns alone she could not get "the" answer. It did not occur to her a *substitution* or a *comparison*, although she had already produced the necessary steps to use any of the two strategies.

$$
\begin{array}{ll}
A + B = 185 & A = 185 - B \\
A - B = 4\cancel{8}7 & B = 185 - A \\
185 - 47 = 148 & 47 \\
\cancel{47} & \\
B + 47 = A &
\end{array}
$$

Nicola D, FM3

Finally, we have Adriana C's (AH7) solution, in which she fails to perceive that letting the same letter to stand for both secret numbers is the main cause of her attempt not working.

$$
\begin{array}{l}
x + x = 185 \\
x - x = 47 \\
2x = 185 \\
-2x = 47 \\
x = \frac{185}{2} = 92{,}5
\end{array}
$$

Não deu certo...!

Adriana C, AH7

As she was writing the first two lines she might well have been aware that the two secret numbers could be different, and was making use of a heavily context-dependent notation (thinking of "a number" and "a[nother] number"), but then she shifts her attention to the written expression and looses control of the process. It is also interesting to notice how she tried to make sense of the second equation

$$x - x = 47$$

by producing

$$-2x = 47$$

instead of accepting the obviously "puzzling"

$$0 = 47$$

Although so evidently distinct in terms of the level of knowledge and technical competence, in those last three scripts one can see the unknown numbers (or parts) being part of the solution process, ie, being assumed as *objects* in the model, as having the same properties of the known ones[34] (***Analiticity***). Also present in all three is a willingness to manipulate *numerical-arithmetical* expressions in order to produce the answer, this manipulation developing within the *Semantical Field of Numerical-arithmetical* expressions[35].

Only three **OKCALC** solutions were produced, two of them of interest to us.

First we have Laura W's (FM3) solution. Her solution to this problem is exactly the same she gave to **Carp1-1** and **Carp1-2** (scripts also shown bellow), and we are led to believe that she actually *modelled back* the set of equations into wooden blocks as in the **Carp** context.

$\frac{185}{2}$. 92½ . $\frac{47}{2}$ 23½   23½ + 92½ 116.
92½ − 23½ 89.
~~N66+89~~ =

Laura W, FM3 — **Sets1-1**

162 ÷ 2 = 81cm . I divided 162 by 2 then
81 − 14 = 67 . added half of 28 (14 onto
81 + 14 = 95 it then took 14 off the
other half.
Small block 67cm
Big block 95cm .

Laura W, FM3 — **Carp1-1**

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[34] At least at a manipulative level.

[35] Actually, Ricardo's and Nicola's solution *could* be entirely justified in terms of ***whole-part*** and ***sharing*** — which nevertheless does not seem to be the case, specially in Ricardo's case. In Adriana's solution, however, we have the expression

$$\mathbf{-2x = 47}$$

which indicates some degree of — if not conscious — *numerical internalism*.

$L + 20 = 162cm$.
20 = 28cm less than the long block
$\frac{162cm}{2} = 81cm$
$\frac{28}{2} = 14\ cm$
95cm + 67cm.
big block 95cm
small block 33½
small block 33½
I divided 162 by 2 then divided 28 by 2 ~~got~~ then I added ½ of 28 to half of 162 then I took the other half of 28 off the other half of 162 then I divide 67 by 2
81 - 28 = 67
Laura W F3A Questions

Laura W, FM3 — **Carp1-2**

Second, we have Joe V's (FM3) solution[36].

I check this by adding 115.5 to 68.5.
115.5
+ 68.5
184.

Now, which are the secret numbers?
(Explain how you solved the problem out and why you did it that way)

184 − 47 = 137 (184 and 47 are the ~~prices~~ number that come of 1st no + 2nd no. = 185, 1st no − 2nd no = 47
no I divide 137 by 2 to find the ~~first no.~~ 2nd no.
2)137 = 68.5
now I add 68.5 (2nd no) to 47 to find the 1st no. 47 + 68.5 = $\boxed{115.5}$ first number $\boxed{68.5}$ 2nd no.

Joe V, FM3

A few points indicate that his is an *non-algebraic* solution and not a non-symbolised *algebraic* solution: *he begins by subtracting 47 from 185*; if the intent of this step was to work out the resulting right-hand side that would result from subtracting the second equation from the first, one has to assume that he did it in order to eliminate the first secret number from the resulting expression. But if this was his intention, why not simply add the two equations, a much simpler procedure by all means? On the other hand, we may see this subtraction as an *evaluation* of the result of taking the excess 47 from the total, so to produce two equal parts, and that he perceived the 47 as an excess of the first number over the second is clear from the fact that near the end of the solution (right before checking his answers up) he says "...I add ...[the] (2nd no) to 47 to find the 1st no.".

[36] As we said before, the fact that he made a numerical mistake was of no importance to us, once the process would lead to a correct answer.

The third solution offers only the calculations and no explanation as to why those steps were chosen.

What emerges clearly from the **WCALC** solutions is that the lack of some kind of written representation seriously hindered the solution process, as those students were trying produce a chain of calculations that made sense and produced an answer. One script is particularly illustrative (Ian C, FM3), who seems to be doing well, only to make a mistake on the last calculation, most probably by judging 69 to be the first and not the second secret number.

$$185 - 47 = 138$$
$$138 \div 2 = 69.$$
$$69 - 47 = 22$$

Ian C, FM3

The Sets 1-3 problem

All the **OKEQT** solutions to this problem used a set of simultaneous equations.

Three of them were solved by a *substitution* method, eg, Daniela V (AH8), in which script we find explicated a very important characteristic of the algebraic method, the need to distinguish different unknowns and parameters from the outset, to assure that the correctness of the derived relationships is kept.

$$\begin{cases} x+3y=185 \rightarrow x=185-3y \\ x-3y=47 \end{cases}$$
$$185-3y-3y=47$$
$$185-6y=47$$
$$-6y=47-185$$
$$-6y=-138$$

$$6y=138$$
$$y=23$$
$$x+3\cdot 23=185$$
$$x+69=185$$
$$x=185-69$$
$$x=116$$

Se um nº é y o outro será x, pq são distintos, substituindo x vou [illegible] mais o resultado da operação temos o valor de y.

Daniela V (AH8)

"If one number is **y** the other will be **x**, because they are distinct..." (beginning of text)

Ten of the sets of simultaneous equations were solved by the *addition* method, and three of those solutions present us with characteristic aspects of *algebraic thinking.*

In Ricardo M's (AH7) solution, the addition of the two equations is justified as he writes down "-3m+3m" and only then simplifies it. This procedure shows the *arithmetical internalism* characteristic of thinking algebraically as it gives the reason for adding the two equations **and** a justification for the addition producing an equation in only one unknown that is completely based on a property of numbers[37].

$$n+3m = 185 \qquad\qquad 1^{\circ}n^{\circ} = 116$$
$$n-3m = 47 \qquad\qquad 2^{\circ}n^{\circ} = 23$$
$$2n-3m+3m = 185+47$$
$$2n = 232$$
$$n = \frac{232}{2} = 116$$

Ricardo M, AH7

Walter R's (AH8) solution exhibits the *method driven internalism* characteristic of thinking algebraically. For no "good" reason he first multiplies the first equation by minus one and only the performs the addition of equations[38]. Nevertheless, the objective of such step is to prepare the set of equations for a subsequent transformation, ie, it is *meaningful* within the *Semantical Field of numbers and arithmetical operations.*

$$\begin{cases} x+3y = 185 \\ x-3y = 47 \end{cases} \qquad y = \frac{138}{6} \qquad x + 69 = 185$$
$$(-1)\begin{cases} -x-3y = -185 \\ x-3y = 47 \end{cases} \qquad \boxed{y = 23} \qquad x = 116$$
$$-6y = -138$$
$$(-1)\ 6y = 138$$

2º 23 e 116 (sistema)

Walter R, AH8

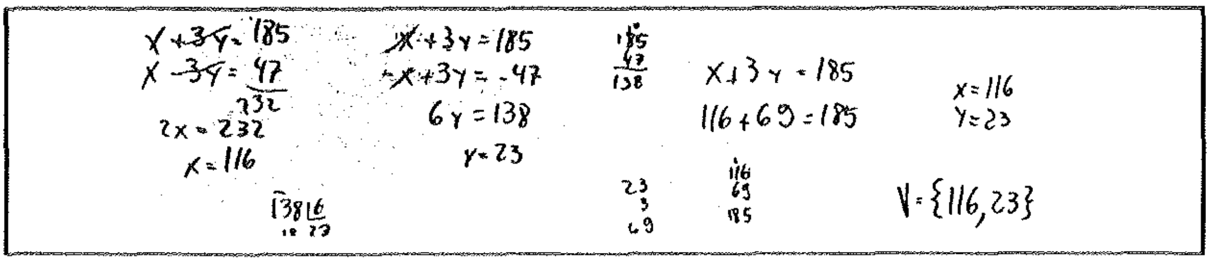
[37]We think that the particular detail of Ricardo writing "2n - 3m + 3m" instead of "2n + 3m - 3m" (the "natural" order, following the order of the equations) shows that he was thinking of the *addition of opposites* property and not of "take away and put back" or "complementing" strategies, the former corresponding to a way of avoiding to write "+3m + (-3m)", a mere symbolic convenience.

[38]The quotes mean that he could have obviously applied the *addition* strategy without this extra step.

Finally, Giuliano G (AH8) sees the generality of the *method of addition* in enabling him to find either of the unknowns from the same set of simultaneous equations by applying the same strategy, and it shows that:

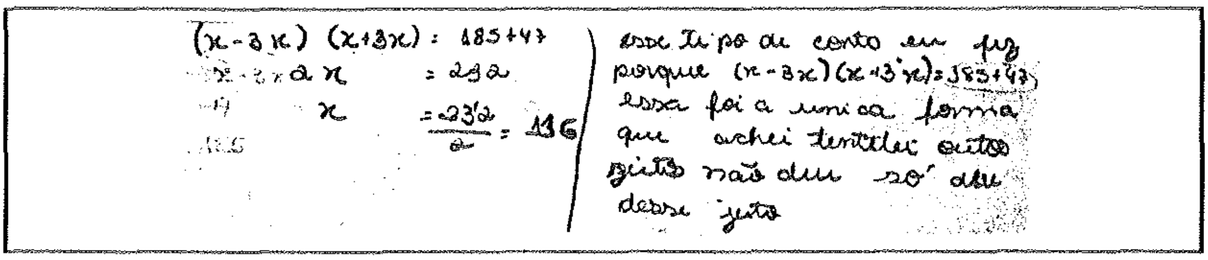
(i) it is the *addition of opposites* that is the centre of his attention (an *arithmetical* property), and,

(ii) although dealing with a numerically specific instance, the generality of the method is clearly expressed even if no "generalised numbers" ("letters") are used for parameters.



Giuliano G, AH8

One of the **WEQT** solutions (Juliana B, AH7) shows one of the possible effects of not distinguishing the two unknowns.



Juliana B, AH7

The result for the first secret number is <u>incidentally</u> correct, given the "friendliness" of the set of simultaneous equations, but she fails to perceive that the second secret number had not yet been determined (also because she does not check the answer against the problem's statement)[39]. It is also interesting that she does not use a "+" sign between the two bracketed expressions on the left-hand side of the equation on the first line, but operates correctly on it, which suggests that the *conjoining* meaning of addition was used

[39] In fact it is not possible to firmly determine whether she did not distinguish the two unknowns at the level of the problem's statement or at a symbolic level, the latter being carried through the remaining steps of her solution process to end with her giving the answer "**The** number is 116" (bottom line at the left).

in "putting the two equations together", rather than a purely *numerical-arithmetical* one. Nevertheless, she was aware that both "conditions" (equations) had to be taken into account, and did not simply substituted **x** for both numbers in one or both equations and proceeded from that to produce the answer, as did Bartira (AH7).

Exemplo:

$1^{\circ}\ n^{\circ} = x = 25$

$2^{\circ}\ n^{\circ} = x + 1 = 26$

$$x + (3.(x+1)\ 185$$
$$x + 3x + 3 = 185$$
$$4x = 185 - 3$$
$$4x = 182$$
$$x = \frac{182}{4}$$

$$x - (3.\ x+1) = 47$$
$$x - (3x+3) = 47$$
$$x - 3x - 3 = 47$$
$$-2x = 47 + 3$$
$$(-1).-2x = 50.(-1)$$
$$2x = -50 \quad x = \frac{-50}{2} = 25$$

Bartira, AH7

Bartira added the extra condition

$$\begin{cases} \textbf{1st number} = x \\ \textbf{2nd number} = x + 1 \end{cases}$$

reducing the problem to one in one unknown only and correctly manipulated the two resulting equations[40]; we want to emphasise that she correctly handled the distribution of **3** over **x+1** even if the latter was not indicated by brackets, and this shows that she was being guided by properties of numbers and also that she was keeping control of the structure of the expressions she was manipulating, even if the notation did not suggest so. Bartira's mistake was at the level of understanding the relationships implied by the problem's statement (a *modelling* mistake), and not at the level of thinking algebraically.

Another **WEQT** solution (Rubens K, AH7) presents the case of manipulation of algebraic expressions being deformed by considerations *external* to the *Semantical Field of numbers and arithmetical operations.*

[40]This is not entirely true, as she makes a mistake on the very last calculation, putting **(-50)/2 = 25**. However, as she did not make any other mistakes in calculations with directed numbers, it might well be that this was not a true error, being instead a deliberate subversion of the usual rules in order to make the result to fit her expectations (for example, that the numbers were positive, an expectation which could have come, for example, from the fact that the answer resulting from the first equation was positive).

$(n1) + (3.n2) = 385$.
$n1 + (3n2) = 385$.
$n + 3n$
$4n = 385$
$n = \frac{385}{4}$ 53+14 25.46, 30 não deu certo

Rubens K, AH7

Rubens begins by deciding to deal with the first equation separately, and correctly identifies two unknowns (**n1** and **n2**). Being unable to proceed from there, he wipes out the distinction in order to reduce the equation to one in one unknown, correctly solves the resulting equation, but fails to go any further, apparently because he could not see how to "revert" the process and go back to the two distinct unknowns.

On the **WCALC** group, the most common error was to take the two conditions given in the problem's statement separately. As one cannot "solve" any of the two equations separately[41], usually this error was followed by the additional error of trying to produce an "answer" by dividing the independent term by **3**, the only other "visible" number in the expressions (Nicola B, FM3).

$\frac{185}{3} = 61{\cdot}2$ [illegible] [illegible] $\frac{47}{3} = 15{\cdot}2$
$61 +$

first no = second no =

Nicola B, FM3

Gurdeep S (FM3), however, goes further, producing a series of calculations that actually result in the correct second secret number.

[41]One could obviously treat each of them as an indeterminate equation in two variables and find some solutions or express a dependence condition explicitly, but it is clear that this procedure was far too sophisticate for those students.

~~185 ÷ 3 = 61.666667 = secret number~~

15.666667

185 ÷ 3 = 61.666667 61.666667

47 ÷ 3 = 15.666667 − 15.666667

46.000000

÷ 2 = 23 =

Gurdeep S, FM3

His procedure could be seen as corresponding to the algebraic procedure

$$\begin{cases} x + 3y = 185 & (\div 3) \\ x - 3y = 47 & (\div 3) \end{cases}$$

$$\begin{cases} \dfrac{x}{3} + y = \dfrac{185}{3} & (\mathrm{I}) \\ \dfrac{x}{3} - y = \dfrac{47}{3} & (\mathrm{II}) \end{cases}$$

$$2y = \frac{185}{3} - \frac{47}{3} \qquad (\mathrm{I}) - (\mathrm{II})$$

$$y = \frac{\frac{185}{3} - \frac{47}{3}}{2} = 23$$

Although possible, this interpretation is highly unlikely to be correct because:

(i) to keep control of the solution process is not simple even with the help of algebraic notation; without it, it seems to be at least very hard;

(ii) if Gurdeep had in mind the *subtraction of equations* strategy, he would have probably applied it directly, without going through the step of dividing both equations by 3.

We offer the following alternative interpretation. Gurdeep begins by dealing with the two relationships separately, and "ignoring" the first secret number he produces the second secret number from each equation[42]. Realizing that he had produced two distinct values, he then tries to make sense of and to coordinate the two pieces of information. We believe that he tried to do so by "averaging" the two values he had obtained.

[42] This initial part of our interpretation is supported by the fact that on the first line of his script he wrote "185/3 = 61.6666667 = secret number"

Only one **OKCALC** solution was produced (David W, FM3), and it is clearly *non-algebraic*, most probably supported by the imagery of a number line (see fig. CCS 2).

The first number is 116, the second is 23. To do this, I found the middle number in between 185 and 47. To do this, I found the difference between the two and halved it. This gave me the first number. To get the second, I found the difference between the first no. and either 185 and 47. This was 69. I then divided this by 3, to get the second number.

David W,FM3

The text in David's script has to be in a sense "decoded", because it does not literally correspond to his solution.

- He first says that he "...found the middle number in between 185 and 47. "*To do this I found the difference between 185 and 47.* This gave me the first number." It is clear that it is not the difference between 185 and 47 that produced the middle number, which he correctly gives as 116. Rather, he found the difference between 185 and 47 (138), divided it by two (69) and added the result to 47 (all three calculations at the left of the script). In relation to the diagram in fig CCS 2, this corresponds to finding the distance between the two extremes **A** and **B**, halving it and adding this to **A** to produce the point **M**.
- He the says that "...To get the second, I found the difference between the first number and **either 185 and 47** [our emphasis]...", a step that clearly corresponds to finding the <u>distance</u> between **A** and **M** or between **M** and **B**.
- Finally, he divides the result by **3** to find the second number, as the distance between the first number and either 185 or 47 corresponds to three times the second number.

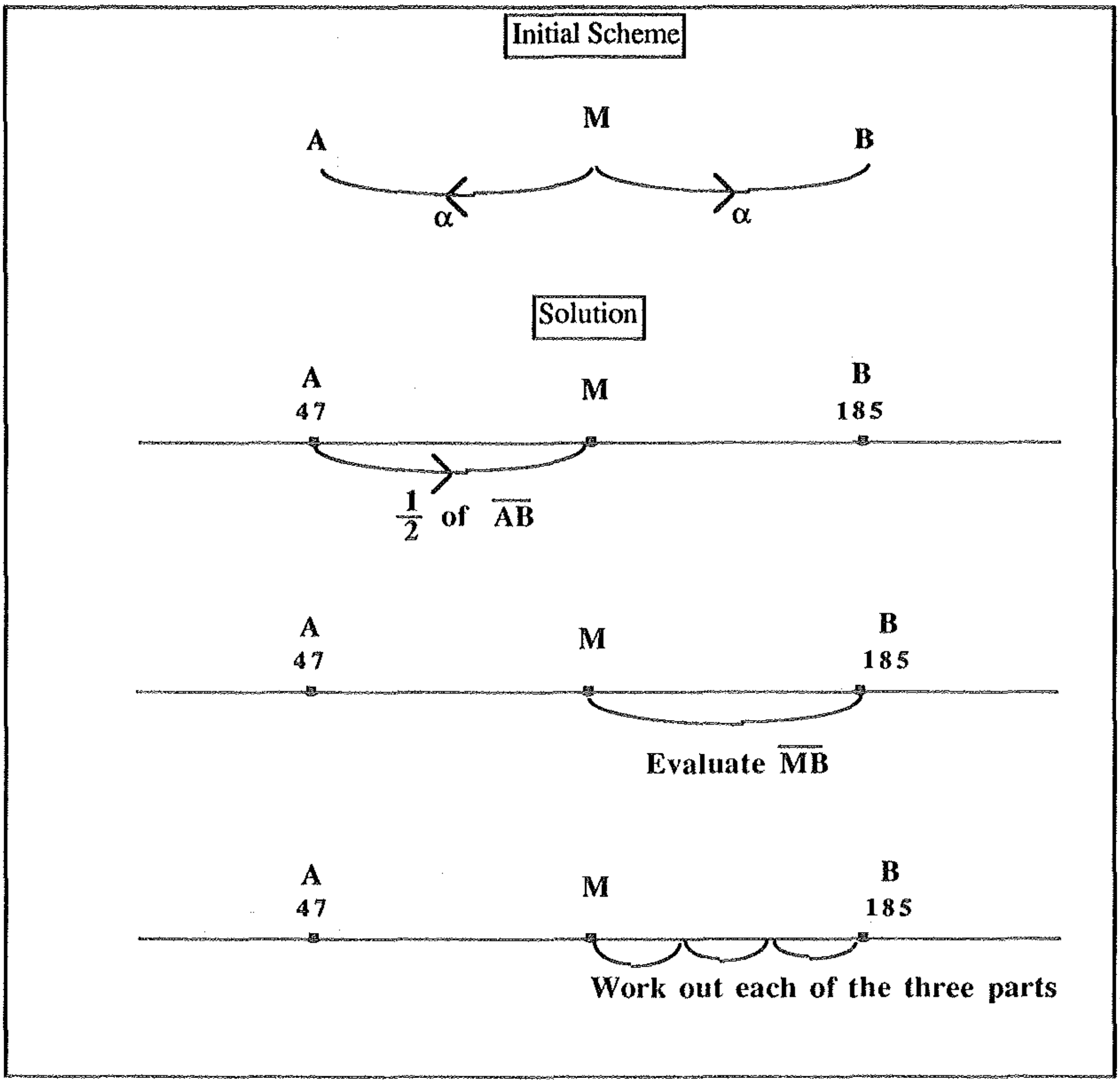


fig. CCS 2

David's solution is *synthetical.* It always proceeds by using the known values to calculate new values until he finally reaches the required answer. It is reasonable to suppose — although no explicit indication exists in the script — that the structuring of the problem itself never involved assuming the unknown values as known in order to guide the process of solution. Given David's description of his solution process, we believe he began by reasoning that the first number was a sort of "centre" from which the same amount was taken from and added to (or, in the context of the geometrical imagery, two points taken, to the right and left of the "centre", and at equal distances — see the "Initial Scheme" on fig CCS 2); from this model it is possible to envisage the necessary steps to produce the answer without any *analytic* reasoning being involved[43]. A second point of interest is that he did not realise that he had already worked out the difference between the middle and extreme points, and recalculates it as **185-116**; the relevance of this point is that it suggests that at each step a new model was produced and then manipulated according to what was seen as relevant in *that* model, and that previous *evaluations* and manipulations were not necessarily seen as "belonging to" the most recent model. Finally, it is worth to remark that he produce a literal representation of the problem's statement (upper left corner of script), that although incorrect — it uses x for both unknowns — might have been important in suggesting the geometrical model by compacting the problem's statement.

The Carp 1-1 problem

WCALC solutions were mostly of two types.

Five students misread the problem's statement and assumed that the length of the shorter block was 28cm, consequently getting the length of the longer block by simply subtracting 28cm from the total 162cm. It is almost certain that this type of mistake arose from a poor reading of the problem's statement, but it has to be pointed out that it was favoured by the actual typing of the questions, which in both Brazilian and English versions — especially the latter — might suggest the mistaken interpretation to a reader more inclined to "quickly inferring."

Twelve students, however, used a more complete — although incorrect — approach (eg, Fabiola AH7). Those students used a "÷2, +28, -28" strategy that many students had used in the exploratory study. This mistaken procedure is certainly due to a failure to perceive that taking 28cm from one of the halves automatically makes the difference between the two measures to be 28cm, but while satisfying the "difference of lengths" requirement, it alters the total length. Those students perceived this unwanted effect and corrected it by adding to the other half the 28cm that had been taken away to produce the shorter block. This step, in its turn, if adjusts the values to satisfy the "total length" requirement, alters the difference between the blocks, thus producing incorrect answers.

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[43]The only relevant property used is that the middle point is at equal distances from the extremes.

g+p = 162cm
p 28cm - que o g

162 | 2
02 81
0

81 − 28 = 53

81 + 28 = 109

O 81cm seria se os dois blocos fossem iguais + o pequeno é 28cm menor que o grande (81-28) e o que der é o pequeno. Depois é só fazer (81+28) que é o grande.

g - 109cm
p - 53cm

109 + 53 = 162

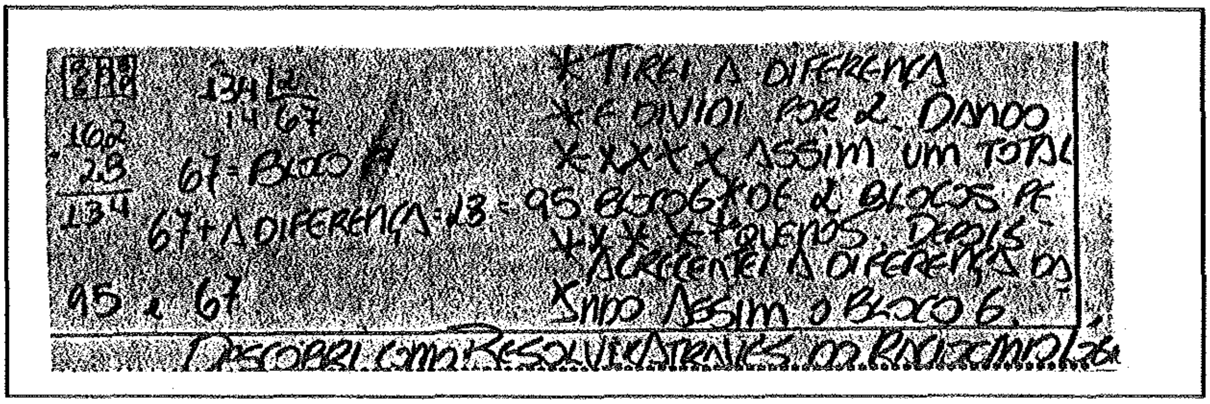
Fabiola, AH7

"81cm would be if both blocks were equal, but the small is 28cm smaller than the big one (81-28) and what you get is the small. Then it is only to do (81+28) and that's the big [block]."

At the root of this kind of mistake is a characteristic of many of the *non-algebraic* solutions presented, and that we have already examined on the last paragraph of the last sub-section, namely the fact that at each step of the solution process a new model is produced — representing or not a correct derivation from the previous models — and it is the most recent model that is manipulated *according to what is perceived as relevant and required in relation to **this** model*; *each step is **locally** meaningful*. The result is a step-by-step solution in the sense that the goals and the means to achieve them might be constantly changing, sometimes resulting in a loss of overall control of the solution process or in a deterioration of the original conditions and requirements through overall inadequate transformations of the intervening models.

The **OKEQT** solutions offer a variety of approaches.

The most common strategy was to *take away* 28cm from the total, so to produce two short blocks, and divide the result of the subtraction by two to obtain the length of the short block; then add 28cm to the length of the short block to obtain the long one (eg, Bruno N, AH8).



Bruno N, AH8

"I removed the difference and divided by 2, resulting in a **total of two short blocks** [our emphasis]. Then I appended the difference [,] resulting in the big block. I found out how to solve it by logical reasoning."

Bruno's solution is a very clear and well explained instance of the use of this strategy, including a diagram that is enough to guide the whole solution process. Some aspects of his solution are of extreme interest to us. The presence of the diagram assures us that the word "tirei", that in Portuguese could also mean "subtracted", is used in the sense of "removed". Moreover, he says that the division resulted in "...a total of two short blocks...", clearly corresponding to a "cut" followed by a division to evaluate the lengths of the two resulting halves. Finally, the word "acrescentar", that in Portuguese might also be interpreted as "adding", has to be interpreted here as meaning "appending", in agreement with the clear-cut indications of the rest of the script. The *objects* being manipulated in Bruno's solution are objects of the context, and the choice of operations is subordinated to the need to *evaluate* measures; moreover, his solution is totally *synthetical*, working from known *objects* to produce other *objects* that are shown to satisfy the required conditions. As in David W's solution to **Sets1-3**, Bruno's solutions never deals directly with as yet unknown *parts*.

Hannah G's (FM3) solution is very similar to Bruno's, but instead of "cutting" the difference to make two short bars, she adds the difference to the total, *pretending there were two long blocks*, showing that hypothetical manipulation of the context of the problem can become a key element in non-algebraic solutions. In Hannah's script one can also see the extent to which the choice of operations is subordinated to the manipulation of the non-numerical model ("I did this to ***find out how much they measured if they were the same length.***")

162, 28

$$\begin{array}{r} 162 \\ -\ 28 \\ \hline 134 \end{array}$$

$$\begin{array}{r} 162 \\ +\ 28 \\ \hline \cancel{190} \end{array}$$ ← I did this to find out how much they measured if they were the same length.

$2\overline{)190}$ = 095 ← is how ~~much~~ long they'd be each

one block = 95   95 − 28 = 67   2nd block = 67 (95+67=162)

Hannah G, FM3

Two other **OKCALC** solutions are worth examining, both using a "÷2, +14, -14" strategy.

We think that Joe V (FM3) decided that he had to add and subtract 14, and not 28, based on his perception — probably due to the expression on the second line — that the 28cm "in excess" on the long block had also been divided in two, an interpretation that is supported by him writing

$$\mathbf{81 + \frac{28}{2}}$$

before writing

$$\mathbf{81 + 14}$$

which indicates that the former expression carried with it something important enough to be made explicit.

first of all you divide 162 by 2

162÷2 = 81 this is the $\frac{\text{long blo.} + \text{short blo.}}{2}$

now as a short block measures 28cm less than a long block

$81 + \frac{28}{2}$ = must equal the length of the long block

81 + 14 = $\boxed{95cm}$ long block

therefor a short block's length is $\boxed{67.}$ short block

check 95+67=162.

Joe V, FM3

On Ricardo G's (AH8) script, on the other hand, there is no clue to how he decided to add and subtract 14, but it is his peculiar way of using algebra that we want to examine.

$x + x = 162$ $\quad$ $x = 81 + 14 = 95$

$2x = 162$ $\quad$ $x = 81 - 14 = 67$

$x = \frac{162}{2}$

$x = 81$

Bloco grande 95 cm. Bloco pequeno 67 cm.

Ricardo G, AH8

He clearly begins with the assumption of the blocks being of the same size, and writes down and solves an equation that reflects that; just by looking at the equation one cannot decide whether he was dealing with a *numerical* relationship or simply using the literal notation to describe an *non-algebraic* process. In any case one has to notice that he explicitly deals with the unknown number-measure, ie, this part of the solution process has an *analytic* character. At the following step, where he adds and subtracts 14, it becomes clear that he saw the division by two as producing **two halves** instead of producing **one value**, as each of the two lines begin with **x** (one of the halves) and represent in fact *the transformation of each half (x) into the required blocks*. His is a *non-algebraic* solution "dressed" in algebraic notation[44].

Tatiane R's (AH7) solution is another instance of a *non-algebraic* solution "dressed" in algebraic notation, but it seems much closer to a true algebraic solution than Ricardo's, as the model used to set the equation takes aboard — as unknowns —the lengths to be determined, as opposed to Ricardo's solution (see note 20), and she produces an equation that directly and simply represents the problem's statement.

[44]Although it is obvious that one cannot be totally sure that the equation was not seen as a *numerical* expression, and that subsequently a shift in the meaning of **x** occurred, we think that in the face of the model he used to set the equation — with **x** representing none of the unknown lengths — *together with the use of x in the remaining two lines*, we must conclude for the "non-algebraic" interpretation.

bloco pequeno = x
$2x + 28 = 162$
$2x = 162 - 28$
$2x = 134$
$x = 67$ cm
bloco grande = 67 + 28 = 95.

Os dois blocos juntos = 162
Só que se eu tirar o pedaço de bloco que tem a mais do que no bloco pequeno então era a mesma coisa que dois blocos pequenos mais o pedaço a mais.

Tatiane R (AH7)

"The two blocks together = 162cm

But if I remove the bit of block that is in excess in relation to the small block, then it is the same as two small blocks plus the extra bit."

Her explanation however, fully reveals that throughout the process of solving the equation she was being guided by — or at least constantly checking for meaning against — the manipulation of a model that took the objects of the context as *objects*, an *non-algebraic* model. The decisive detail in the text is when she says that "it is the same as two small blocks ***plus the extra bit***," showing that the solution process was in fact guided by a *composition-decomposition of parts* process.

In the **OKEQT** group of solutions, a number of points arise.

Alessandra O (AH8) produces a substitution in the context of the set of equations, while Andrea M (AH8) produces a *direct non-algebraic substitution*, to solve the problem from a single equation.

$$\begin{cases} x + y = 162 \rightarrow 95 + y = 162 \\ x - 28 = y \qquad y = 67 \end{cases}$$

$x + x - 28 = 162$
$2x = 190$
$x = 95$

R: o Bloco grande tem 95 cm de comprimento e o pequeno 67 cm. Resolvi através de um sistema

Alessandra O, AH8

$x + x + 28 = 162$
$2x = 162 - 28$
$2x = 134$
$x = 67$

$67 + 28 = 95$

→ o x será o número do bloco pequeno, já que eu não sei as medidas completas, mas sei o número de "comparação" de um para o outro. Eu faço o mesmo processo se eu tivesse as medidas completas: somo. A soma é feita normalmente somo separado os x's e os números. Depois separo o x de um lado e os números no outro. se tiver ainda algum número junto ao x passo para o outro lado com a operação inversa.

Andrea M (AH8)

"x will be the *number* of the small block, as I don't know the complete measures but known the number of "comparison" of one to the other. I do the same process as if I had the complete measures: add. The sum is done normally [,] I add separately the numbers and the x's. Then I separate x to one side and the numbers to the other. If there still is some number with x, I move it to the other side [,] with the inverse operation." (our emphasis)

Andrea's solution, moreover, provides a clear statement of:

(i) the *analiticity* of her reasoning, by saying "I do the same process as if I had the complete measures: add.";

(ii) the *arithmeticity* of her reasoning, by saying that "x will be the ***number*** of the small block..." and treating numerically the setting of the equation.

Marília M's (AH8) and Rogério C's (AH8) solutions exhibit an important feature of thinking algebraically, the use of *normal forms* of numerical-arithmetical expressions.

→ Através de um Sistema

$\begin{cases} x + y = 162 \\ x - 28 = y \end{cases} \Rightarrow 95 + y = 162$

$y = 162 - 95$

$y = 67$

$\begin{cases} x + y = 162 \\ x - y = 28 \end{cases}$

$2x = 190$

$x = 95$

$162 + 28 = 190$

$162 - 95 = 067$

$190 \div 2 = 95$

Bloco grande = 95 cm

Bloco pequeno = 67 cm

Marília M, AH8

$$\begin{cases} x + y = 162 \\ x - y = 28 \end{cases} \Rightarrow 95 + y = 162$$

$$+y = 162 - 95$$

$$2x = +190 \qquad +y = +67\text{cm}$$

$$x = \frac{+190}{2}$$

$$x = 95\text{cm}$$

R: O comprimento do bloco grande é 95cm o comprimento do bloco pequeno é 67 cm.

explicação: resolvi pelo processo de sistema

Rogério C, AH8

In Marília's case, the *normal form* is produced at the *algebraic* level, by manipulating the second equation

$$x - 28 = y$$

to produce

$$x - y = 28$$

while in Rogério's case the *normal form* is directly produced by interpreting — and representing — the fact that one of the blocks is 28cm longer as meaning that the difference of their lengths is 28cm[45].

The Carp 1-2 problem

An undesirable and unexpected effect appeared in relation to this problem, with nine students solving **Carp1-2** as if it were **Carp1-1**, ie, only one short block had been mentioned in the problem's statement. We are led to believe that those students had already been presented with **Carp1-1** on the first session, and when they saw **Carp1-2** they did not bother to read the statement, as both the drawing and the first sentence are the same in both problems' statements, a flaw in the design of the tests[46]. Also, five students solved the problem assuming that 28cm was the length of two short blocks; this mistake had already been identified in the solutions to **Carp1-1**, and here again it might have been urged by the unfortunate choice of line break for the text.

Other **WCALC** solutions reveal some difficulties caused by the increase in complexity in relation to **Carp1-1**.

Ricardo B (AH7) applies a "generalised" version of the "÷2, -28, +28" that was examined in relation to **Carp1-1**.

[45] This type interpretation was in fact very rare in all the problems in all groups.
[46] Our original intention was to cause the two problems to be seen as much as possible as very similar.

O bloco grande mede 82cm e o bloco pequeno mede 40 cm.

162 ÷ 3 = 54; 54 + 28 = 82 cm

162 − 82 = 80; 80 ÷ 2 = 40

80 + 40 + 40 = $\boxed{162}$

São 3 blocos de madeira então eu dividi o total dos comprimentos e coloquei mais 28cm. depois eu subtraí como você esta vendo acima.

Ricardo B (AH7)

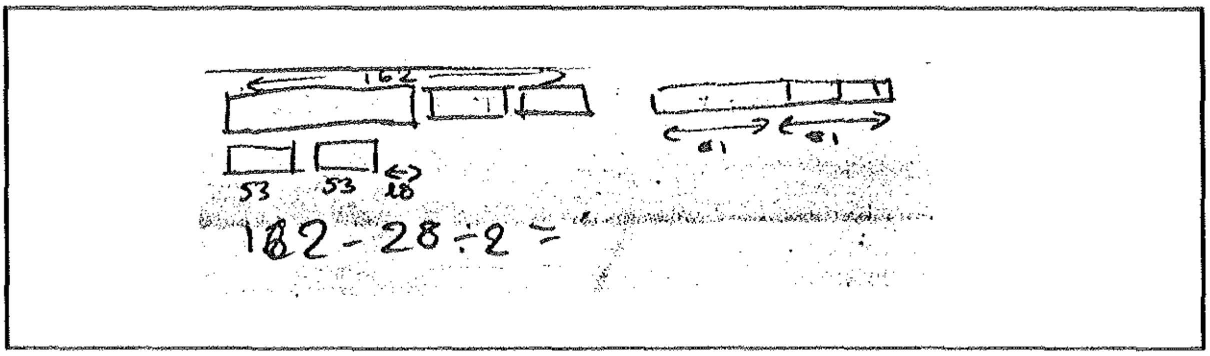
"There are three wooden blocks, so I divided the total length and put another 28cm. then I subtracted as you can see above."

As a result of the increased complexity, Ricardo fails to perceive that the 28cm he adds to one of the parts produced by the *cut-division* makes the long block 28cm longer than **each** of the other ones, but at this stage the two short blocks put together are in fact 26cm longer than the long block[47]. A very odd shift now takes place, as to work out the length of the short blocks he subtracts the now known length of the long block from the total length, and divides the result by two to obtain the length of each short block; it should be immediately clear, as he obtains 80cm for two short blocks that something went wrong, as the difference is only 2cm. We think that this fact was not enough to trigger a revision of the previous working exactly because at that point the model he was working with included only the "total" and the "two short blocks" conditions, but not the "difference" condition; as it had happened with the solutions to **Carp1-1** mentioned earlier in this paragraph, each step resulted in a new model that was then manipulated anew, with the product of previous manipulations not always being taken into consideration[48].

Helen R (FM3) produces a very good diagrammatic representation of the problem (except that the diagram on the right is not correct because it includes the "extra" 28cm in the total as a separate bit), a representation that would almost certainly lead to a correct solution in **Carp1-1,** but fails to draw further information from it and fails to manipulate it into a more *informative* diagram, which suggests that the need to deal with the two short blocks as one single *object* functioned as an obstacle that was not overcome by her.

[47]It is legitimate **at this point** to assume that the two remaining blocks are the two short blocks, as Ricardo's rationale for dividing by 3 is that there are three blocks.

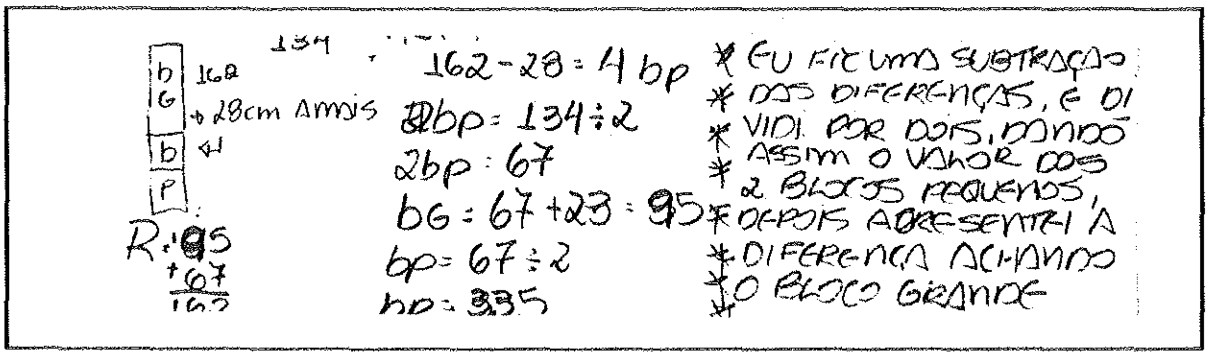
[48]As in the total disregard for the two 54cm bits that *ought* to correspond to the two short blocks — if not immediately, after some possible adjusting steps. Instead he shifts to the model "I know the total length of a long plus two short blocks, and I know the length of the long one, so..."



Helen R, FM3

The **OKCALC** solutions to this problem underline and clarify several relevant aspects of *non-algebraic* solutions.

Bruno N's (AH8) solution[49] shows the way in which a diagram is used to provide a simplified representation of the problem's statement, mixing a *whole-part* figure to represent the first condition, with an added verbal remark ("28cm more") to represent the second condition. It is clear that this diagram guides the solution process, as the labels used in it for the long and short blocks are used throughout, and the first line in the sequence of equalities indicates — by having the numerical calculation on the left-hand side and the part that its result measures on the right-hand side— that the numerical calculations are used to *evaluate* the measures of parts according to the manipulation of the *whole-part* model.



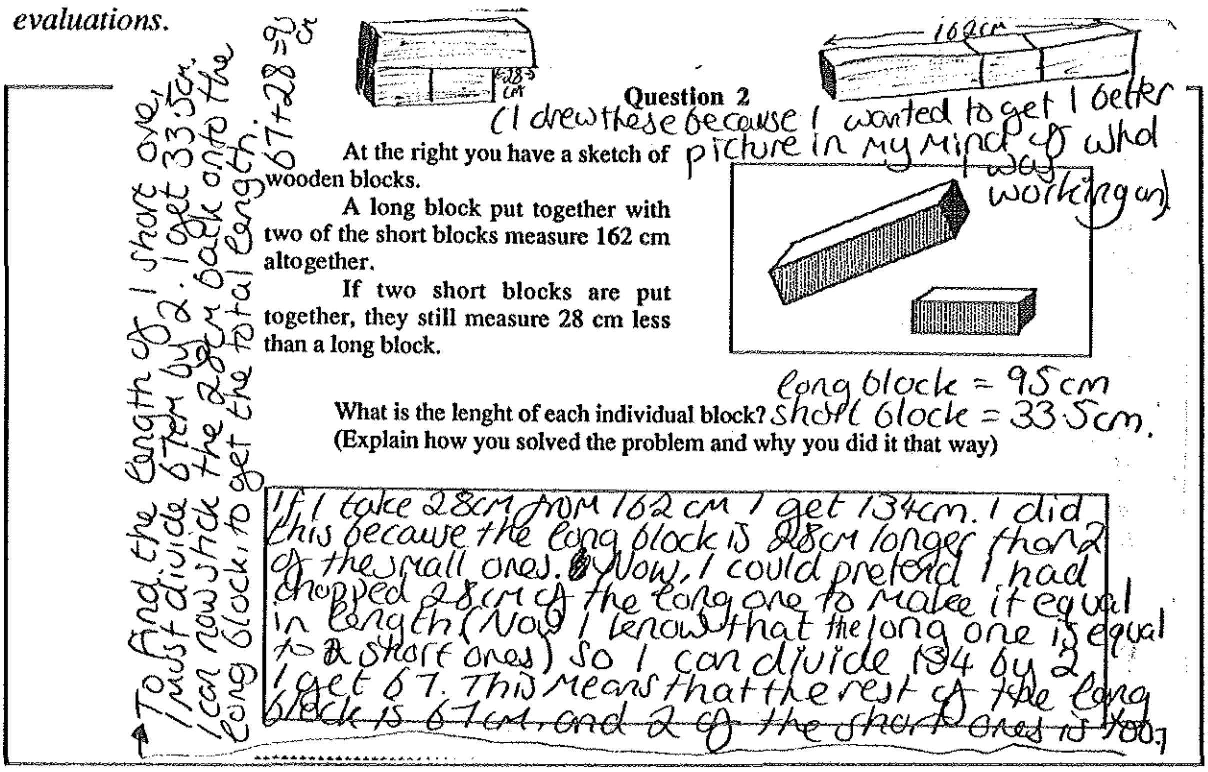
Bruno N, AH8

Elizabeth W (FM3) provided us with what is probably the clearest example of an *non-algebraic* solution among all scripts we examined.

First, because she makes it explicit that the figures she draws at the top are used to guide the solution process. Second, because she always describe the manipulative steps

[49]The text to the right does not add anything that is not already evident in the rest of the script, and for this reason is not translated.

that justify the choice of operations to be performed on the measures to *evaluate* other parts, eg, "...I could pretend I had ***chopped 28cm from the long one...***", and "I can now ***stick the 28cm back into the long block...***". Moreover, in her solution there is a *transformation* of the problem when she reduces it to one where a long block measures the same as two short blocks. This strategy is different from taking the difference away to be left with four short blocks, as it actually establishes a new variable and a new relationship, the shortened long block becoming "<u>the</u>" long block. Her solution is throughout well controlled and *synthetical*, and above all it shows that *verbal language* is totally adequate to describe the hypothetical assumptions and the transformations that support the choice of operations, while standard written arithmetical statements take care of describing the *evaluations*.


Question 2

(I drew these because I wanted to get I better picture in my mind of what I was working on)

At the right you have a sketch of wooden blocks.

A long block put together with two of the short blocks measure 162 cm altogether.

If two short blocks are put together, they still measure 28 cm less than a long block.

What is the lenght of each individual block? long block = 95cm short block = 33.5cm.

(Explain how you solved the problem and why you did it that way)

If I take 28cm from 162 cm I get 134cm. I did this because the long block is 28cm longer than 2 of the small ones. Now, I could pretend I had chopped 28cm of the long one to make it equal in length (Now I know that the long one is equal to 2 short ones) so I can divide 134 by 2 I get 67. This means that the rest of the long ~~block is 67cm, and 2 of the short ones is 100~~

To find the length of 1 short one, I must divide 67cm by 2. I get 33.5cm. I can now stick the 28cm back onto the long block, to get the total length. 67+28 = 95

Elizabeth W, FM3

In Matthew K's (FM3) script also we find a solution process that is typically *non-algebraic*, with the 28cm taken as a separate bit that can be *appended* to the combination of one long and two short blocks, the arithmetical operations being performed to *evaluate* lengths. It is also distinctively *synthetical*.

long block = 95 cm   Short block = 33.5

To work it out. I added 28 cm to 162 to find the 2x long ÷ by two to find 1x long − 28 to find 2x Short ÷ 2 to find 1x short

Matthew K, FM3

Finally, we examine Joe V's (FM3) solution , which uses literal notation ("...a little formula...," as he calls it) but is guided by the manipulation os a *whole-part* model.

First I do a little formula
n = short block, x = long block
n+n+x = 162
now I take the 28cm from 162cm so that the answer is n4
162 − 28 = 134
to find out n I need to divide 134 by 4   4/134 33.5 a small block is $\boxed{33.5cm}$ long

Joe V, FM3

On the second line he writes

$$\mathbf{n + n + x = 162}$$

his "formula", but it is not a *numerical* one, as one gathers from the subsequent manipulation of the model it is intended to represent. Instead, the "+" sign means the *conjoining* "and", and the "=" sign denotes "measures" — acting as a *value label,* as we saw on page.... This interpretation becomes more clear when Joe "...take[s] the 28cm from 162cm so that the answer is n^4 " — in which he obviously meant 4n; the subtraction 162-28 (an *evaluation*) is different in nature from the action that produces the "4n" (a *decomposition*) corresponding to its result[50]. Although apparently it is an *analytic* model, in fact it is not, because the parts of unknown measure are not there to be directly manipulated, but to provide the *whole-part* structure and allow him to visualise a sequence of decompositions, compositions and correspondent *evaluations* that will lead to the answer.

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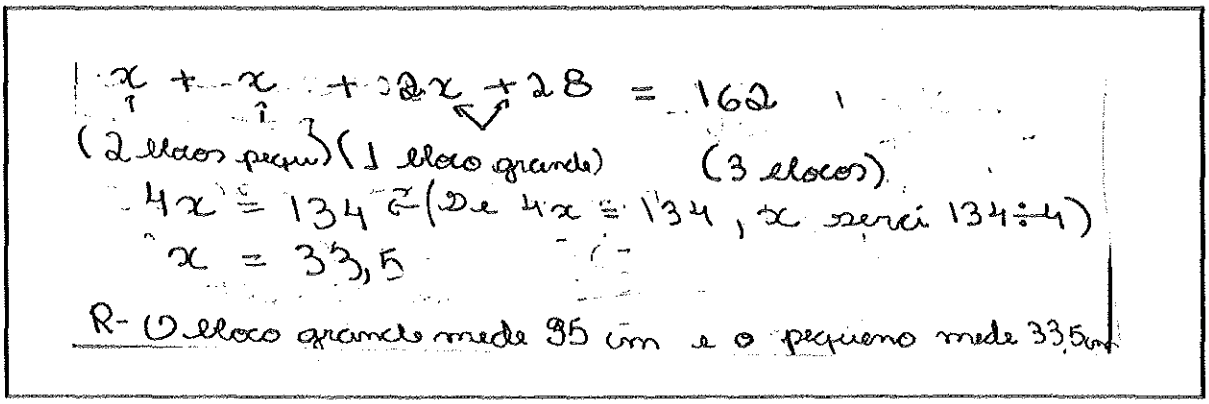
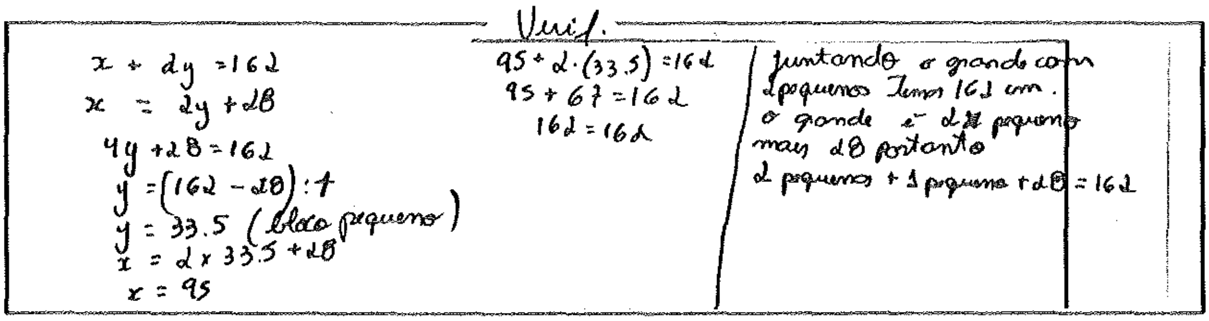
[50]We think it is telling that Joe states the *decomposition* — with its outcome — as a separate and prior step from the actual calculation.

As with the **OKEQT** solutions to **Carp1-1**, we had for **Carp1-2** both cases of a model with a single equation in one variable being produced through a *direct non-algebraic substitution* (eg, Laura G, AH7) and of a model with a set of simultaneous equations being initially produced and from there a substitution that reduces the set of equations to a single equation in one variable (Mairê M, AH8[51]).



Laura G, AH7

"(2 short blocks) (1 big block) (3 blocks)"



Mairê M, AH8

One last **OKEQT** deserves examination. Tatiane R (AH7) first solves the problem with equations (left), with a peculiar use of indexed x's, possibly meaning that she saw the two short blocks in the second line as distinct[52] from those in the first line; the distinction is finally blurred on the fourth line, and the solution correctly completed[53]. On the verbal explanation, however, she shows an understanding of the *back-interpretation* of the

[51] The text at the right of the script is a restatement of the problem's statement, and thus was not translated.

[52] Physically distinct; some *other* blocks.

[53] Although there is a mistake in the subtraction, the solution is considered correct, following our criteria of prioritising the overall correctness of the procedure over the actual calculations.

algebraic procedure in terms of the problem's context that is mistaken ("...when the three [blocks] are equal one has only to divide by the sum that made the three equal"). Had she followed the image of three equal blocks, she would have made a mistake, and this strongly highlights that by focusing the solution process on the *method* and by keeping it *internal*, algebraic thinking provides a powerful way of keeping correct control of it.

Bloco grande = 2x + 28 = 100 2 Blocos pequenos x+x = 72 2x+28+x+x = 162 4x = 162−28 4x = 144 x = 36	1 Bloco grande é igual a 2 blocos pequenos + 28 cm então nós temos que somar os 3 blocos, que é igual 162 desses 162 nós temos que tirar a diferença entre os três blocos. Quando os três ficarem ~~iguais é só dividir pela soma~~ que deu os três iguais

Tatiane R, AH7

Two **WEQT** solutions present two distinct — but both critical — aspects of using algebraic models to solve problems.

Mariana O (AH8) starts by setting a correct single equation in one variable — a *direct substitution* — and correctly solves it for **x** to determine the length of the short block.

$(2x+28)+2x = 162$ $4x = 134$ $\boxed{x = 33,5}$ $x+14 = 47,5$ O grande 47,5 o pequeno 33,5	Coloquei o nome de x no pequeno, e se o maior é 14 cm a mais ele é x+14.

Mariana O, AH8

"I put the name of x on the small, and if the larger is 14cm more, it is x+14"

Having already correctly recognised and used the relationship between the lengths of the long and short blocks, she then shifts to another model and this produces the error. The model she shifts to seems to be related to the "÷2, +14, -14, ÷2" approach[54], which

[54] An extension of the approach of dividing the total in two parts and then adding 14cm to one of them and subtracting 14cm from the other one to produce the required lengths.

nevertheless is not correctly interpreted by her, producing the misunderstanding that the longer block is 14 cm longer than each of the short ones[55]. Mariana correctly solved **Carp1-1** using an equation, and we are led to think that the increase in complexity was at least partially responsible for the lack of appropriate control. The crucial point, however, is that the shift to a distinct — although potentially correct — model produced an error, and this indicates the extent to which an algebraic approach depend on keeping the solution within the boundaries of the initially set equations, as the *arithmetical internalism* characteristic of *algebraic thinking* involves a shift away from the *Semantical Field of the Wooden Blocks*, and any new relationship introduced during the process of solution would have to be double checked, first within that *Semantical Field* — to assure that it correctly models the problem's statement — but also against the initial algebraic model, to guarantee, for example, that the unknowns used are in correct correspondence. Marina's lack of perception that the resulting length of the long block is not 28cm greater than the length of the short ones — let alone 28cm longer than **two** of them put together — is remarkable.

The second **WEQT** script we want to examine is Marcel S's (AH8).

$$\begin{cases} x = 2x + 28 \\ x + 2x + 28 = 162 \end{cases}$$

$2x + 28 + 2(2x + 28) = 162$

$2x + 28 + 4x + 56 = 162$

$6x = 162 - 28 - 56$

$6x = 78$

$x = \frac{78}{6}$

$\cancel{x} = 13$

$2x + 28$

$2 \cdot 13 + 28$

$26 + 28$

54

Eu resolvi por ter aprendido o sistema. Eu descobri porque quando eu estava ~~tentand~~ aprendendo o sistema eu fiz varios exercícios parecidos com este

O pequeno mede 13 e o grande 54

Marcel S, AH8

This script shows how deeply an algebraic solution can be guided by the meaningfulness of transformation strategies rather than by any other considerations, ie, how strong a factor the method can become. Marcel's solution has several errors. The first is the failure to distinguish the two unknowns *notationally*, a mistake that we have already examined. Also, the second equation of the bracketed set (top-left) does not model the problem's statement correctly, not even allowing for the interpretation — derived from the first equation — that **x** alone represents the long bar and **x** in "2x" represents a short bar. Finally, when he "substitutes" in the second equation the "value" of the left-hand side **x**, he

[55]She might have reasoned that if the long block is 28cm longer than **two** short blocks, it is 14cm longer than **one** short block.

"omits" the 28 that is immediately to the left of the equal sign on the second equation. Nevertheless, he **does** produce a substitution, one that might seem absurd as he had not one, but two equations in one variable that he could easily solve — as he does with the equation resulting from the faulty substitution — and this indicates that although he did not distinguish the two unknowns *notationally*, he apparently did it *semantically*. Moreover, it might be that the 28 was "missed" because in the *Semantical Field* within which Marcel was operating, it was meaningful only when added to the "2x".

The Choc problem

In previous passages, we have already analysed some of the difficulties caused by the use of context-dependent or loose notation. Two attempted solutions to this problem suffer from such shortcomings, but the outcome — although incorrect in both cases — is quite different. Both Tathy G (AH8) and Daniela V (AH8) use the notation "x + 3" for a box and three spare bars, and "x - 3" for a box with three bars missing.

$$\begin{array}{r} 966 \\ -\ 714 \\ \hline 252 \end{array}$$

$(x+3) = 966$

$(x-3) = 714$

$x+3 = 966$

$x = 966-3$

$x = 963$

$x-3 = 714$

$x = 714+3$

$x = 717$

Tathy G, AH8

Se uma caixa x+3 (mais três chocolates pequenos) = (custam) 966, o chocolate custa o preço de todo ÷ por 3, ou seja $x = \frac{966}{3} = 322$

$x-3 = 714$

$x = 717$

Pois ajuntamos os três chocolates que faltavam.

Daniela V, AH8

"if one box **x** + 3 (plus three spare bars) = (cost) 966, a bar costs the price of all of them ÷ by 3, that is, **x** = $\frac{966}{3}$ = 322.

Because we add the three bars that were missing

Tathy treats the two resulting equations separately, and abandons the attempt when she gets different values for **x**, both equations being correctly solved. On Tathy's solution there is a shift into a *Numerical Semantical Field* immediately after the equations being produced, and this results in the variable "chocolate bar" being simply overlooked and not considered at all after that.

Daniela, on the other hand, stays within the *Semantical Field of the Chocolate Boxes* even after writing — and carefully explaining — the expression "**x** + 3". She then interprets the situation as meaning that the total price corresponds to the 3 spare bars — disregarding the full box — and divides 966 by 3 to obtain the price of a single bar[56]. However, when she uses the same kind of notation to express the second combination, the strategy does not apply any longer, because it makes no sense to think of sharing the total by what is not. It is only then that she tries to make a new sense of the expression and shifts into a numerical-arithmetical interpretation and correctly solves the equation — as meaningless as it can be in regard to the problem's statement. When she tries to justify the shifted procedure, she says "Because we add the three bars that were missing"; there is a clear disturbance in the meaning of the 714.

Nine students produced a value for the price of a chocolate bar by dividing the difference between the two combinations of box and bars by 3, **WCALC** solutions. The root of this mistake is probably similar to what caused the shift in Daniela's solution: those students knew that the difference in price corresponded to a difference in the number of bars, but considered only the spare bars in the first combination, the bars that "actually" existed. Claire B's (FM3) script is quite clear about this, as she labels the 3 as "...(the number of bars in question)..." Also in Claire's script, we find a forceful example of the subordination of the use of the arithmetical operations to the manipulation of a non-numerical model, as she takes away "...£5.31 from £8.85 to get £3.54..." and from there produces the price of a bar, but "...To check *this* [that the price of £1.18 for a bar is correct] I took £3.54 away from £8.85 to get £5.31." (our emphasis)

[56]We believe that Daniela's flow of thought passed through the feeling that the 3 corresponded to the only thing being actually "counted", "the number of chocolate bars" — forget the "spare" — as the number of bars in a box is unknown and is not mentioned as an element of the problem's statement or question.

First I took £5.31 away from £8.85 to get £3.54. I then devided £3.54 by 3 (the number of bars in the question) to get £1.18. To check this I took £3.54 away from £8.85 to get £5.31. So the cost of a chocolate bar is £1.18.

Claire B, FM3

All but one of the **OKCAL** solutions were of one of two types: (i) putting together the two combinations, with the three spare bars in the first combination "compensating" for the missing ones in the second combination (eg, Clare F, FM3[57]), or (ii) proceeding from the fact the the extra price corresponds to 6 extra bars (eg, Cláudia F, AH7).

~~£8.85~~
~~-5.31~~
~~£3.54~~
£8.85
5.31
£3.54

~~£8.85 - 3.54 = 5.39~~ ~~£5.31 + 3.54 = £8.85~~

8.85 + 5.31 = 14.16

2) 14.16 = 7.80 £7.80

£1.05 3) 1.05 = 35

~~the price~~ The price for 1 box of chocolates is £7.80
The price for 1 bar is 35p

To solve the problem I added the price of both boxes of chocolates together. This would give you the price of 2 boxes of chocolates. This is because if you take the three spare bars and put them in the box with 3 missing it gives you two boxes

Clare F, FM3 (script restored)

[57]In Clare F's solution we have the "compensation" strategy explained in terms of a possible *physical* action, but most students in the **OKCALC** category did not mention this kind of *rationale* explicitly.

caixa = x $\qquad$ x+3 = Cr$966,00 e x−3 = Cr$714,00

Essa caixa com +3 chocolates separados, no final terá 6 chocolates a mais que a outra, porque na outra faltam 3 chocolates e na caixa com +6, ela é completa e ainda tem +3 chocolates.

Cr$966,00
Cr$714,00 −
Cr$252,00 } Preço dos 6 chocolates = diferença entre as caixas

Cr$252,00 | 6
12 $\quad$ Cr$42,00
000

Cláudia F, AH7

"box=x

This box with +3 separate bars, in the end will have 6 bars more than the other one, because in the other 3 bars are missing and the box with +6 is full and has +3 bars.

Price of 6 bars = difference between boxes."

Cláudia uses literal notation, but the intention is clearly descriptive only, as those written expressions are never directly manipulated; instead, the *objects* manipulated are objects of the context, and the model based on which the problem is solved is made up of those objects of the context and and relationships involving them, and perceived properties of both the *objects* and relationships.

The one **OKCALC** solution that does not conform to types (i) and (ii) above is David W's (FM3).

A box of chocolate bars costs £7.08. The price of a single bar is 59p.
First, I worked out the difference between 885 & 531, and halved it. I then added that figure, (177) to 531. This was to find the price of a box of chocolates. To find the price of a single bar, I just divided 177 by 3.

David W, FM3

His solution to **Choc** is absolutely similar to his solution to **Sets1-3**, and as we argued before on page 254, it seems to be based on a model involving points in a number line (as in figure CCS 2). David is one of the very few students that produced solutions that are clearly non-algebraic but using a model that is not built based on the objects of the context. Moreover, the model he employed here and at **Sets1-3** is perfectly general for this class of problems.

One solution stands halfway between algebraic and non-algebraic. Walter R (AH8) says that he "...solved with a *system*[58] to find out the box [sic] and subtracted the 966 by 714 and divided by 6 and found out how much is the bar."

$$x + y = 966$$
$$x - y = 714$$
$$2x = 1670$$
$$x = \frac{1670}{2}$$
$$x = 835$$

a caixa é 835
o chocolate separado é 42

$$\begin{array}{r} 966 \\ -714 \\ \hline 252 \end{array} \quad 252 \div 6 = 42 \quad \begin{array}{r} 240 \\ \hline 12 \end{array}$$

Fiz por sistema para descobrir a caixa e subtrai os 966 por 714 divide por 6 e descobri quanto custa o chocolate separado

Walter R, AH8

When he says that used a set of equations, one has an indication of how he classified what he was dealing with, but at the same time the notation is incomplete and one wonders how he would deal with a problem like "a box and three spare bars,..., a box with **two** bars missing." The fact that he starts afresh to determine the price of a bar, suggests that the he did not perceived the "system" as composed by expressions linking the price of a box and the price of the bars, and we are thus led to believe that he was very much influenced by the *form* of the literal expressions in his choice of method of attack to this first part of the problem.

Only one script actually adds to what we have said so far about **OKEQT** solutions. Giuliano G (AH8) uses absolutely the same method of solution — unique in this group of students — he uses with **Sets1-3**, namely, solving the set of equations twice, once for each unknown, and both by the *addition* method. Moreover, his maturity and confidence with algebraic solutions shows in his use of symbolism: if **y** stands for "(the price of) a bar" **xy** stands naturally for "(the price of) a box of **y**'s", or **x of y**. He is never troubled by this potentially ambiguous notation. Finally, we think it is very significant that from a mature algebraic thinker comes the only script in the whole of this group of problems where the answers are checked against both conditions.

[58] See note 32, p242.

$$\begin{cases} xy \cancel{+3y} = 966 \\ xy \cancel{-3y} = 714 \end{cases}$$

$$\begin{cases} \cancel{xy} + 3y = 966 \\ -\cancel{xy} + 3y = -714 \end{cases}$$

1680

$6y = 252$

$y = 42$

252 | 6
12 | 42

$2xy = 1680$

$xy = 840$

UM CHOCOLATE SEPARADO CUSTA 42 CRUZEIROS

UMA CAIXA DE CHOCOLATE CUSTA 840 CADA.

840 + 3(42)

$840 + 126 = 966$

$840 - 3(42) = 714$

840 − 126

126

714

Giuliano G, AH8

SUMMARY OF FINDINGS AND CONCLUSIONS

The analysis of the responses to the problems in this group threw light on many characteristic aspects of both *algebraic* and *non-algebraic thinking*, but also on the ways in which the two modes interact, and on the modelling processes that develop on the border between *algebraic* and other modes.

The issue around which all the others can be organised, is that of *meaning*. Seen in its broader sense — and we think this is the correct approach here — *meaning* is related to the stipulation of which *elements* are to belong to a model and in which way, ie, how they will relate to other *objects* of the model and how those *objects* can be manipulated, or what *properties* they have; *meaning* is related to the constitution of *objects* from *elements*, and inevitably linked to the perception — by the solver — of what could and should be done in order to solve a problem.

In relation to this group of problems, the clearest instance of different ways of producing *meaning* from the *elements* of a problem comes from the **Choc** problem. While a *substitution* strategy involves a strong shift in *meaning* when performed within the *Semantical Field of the Boxes*, it does not when performed within a *Semantical Field of numbers and arithmetical operations*, as we have already seen. Another very important indication of the effect of the types of *objects* that are constituted — and, of course, of the effect of what the solver sees as meaningful in a problem's statement — is in the fact that many students simply could not make sense of the **Sets** problems; taken as arithmetical relationships, they did not provide them with information on how to solve the problem because *to them* arithmetical relationships cannot be constituted into *objects* and manipulated, being rather a form of descriptive, static statement. The other possibility for solving **Sets** problems, modelling them back into another context, ie, interpreting the numbers as measures and the arithmetical operations as *whole-part* operations (conjoining and separating, for example) was thoroughly ignored by the students (only 12% of FM3 did that in **Sets1-1**; no-one else did it in **Sets1-1**, and no student did it in **Sets1-3**). The fact that many students were able to handle — *non-algebraically* — problems with the same *whole-part* structure, shows that the difficulty was in interpreting the arithmetical statements in *whole-part* terms.

Another key element in the direct manipulation of those relationships in **Sets**, the willingness to incorporate unknown numbers or parts into the model and deal with them as if they were known (ie, a willingness to operate *analytically*), was present in none of the non-algebraic solutions. From the examination of the scripts to the contextualised questions, we learned that the lack of *analiticity* is a **consequence of**, rather than a cause to the use of non-algebraic models. Non-algebraic models involved a separation between the objects to be manipulated and the measures involved in the *evaluation* steps; the transformation of a relationship involving two parts of unknown measure can only be meaningful if it enables an immediate or almost immediate *evaluation*. For example, if one knows that "a long block put together with two short blocks measure 162cm altogether", one can derive that "if from the total one removes the long block one is left with two short blocks". Although in terms of *whole-part* manipulation this is an easy step, it does not entail the immediate *evaluation* of any as yet unknown part and is thus, in itself, meaningless in the context of an *synthetic* solutions[59].

Only one student used a non-algebraic, "decontextualised" model[60]. David W's model is clearly *geometrical*. In many instances we could positively identify *non-algebraic* models through their use of objects of the context as *objects* (eg, "cut the extra bit", "move the extra bars to the other box" or "3 bars, the ones that count"), but even on those non-algebraic solutions where this positive identification was not possible — leaving open the possibility of them using a more general *whole-part* scheme, based on a line-diagram, for example — we almost always found that the models used were constrained by limitations very similar to those in a model based on objects of the context (for example, to take 28cm corresponding to cutting the extra bit, but not add 28cm in a hypothetical move), and this characterises a *non-algebraic* model.

Diagrams were used only with **Carp1-1** and **Carp1-2** problems, supporting our conclusion that non-algebraic solutions were almost always *context-based*, as in those contexts bar and line diagrams belong naturally as schematic representations of block combinations. Also, there were more diagrams with **Carp1-2** than with **Carp1-1**, and we think it was so because the greater complexity of the former made it more difficult to be

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[59]There would also be another difficulty, in this specific case, that the 162cm is a *measure* to the combination of blocks, and only meaningful in this respect.
[60]That means, out of the original context of the problem

handled without the aid of a representation on paper. The lack of written representation resulted many times in the solver loosing track of the unknowns or of the solution process[61].

In most of the solutions using equations we could reasonably establish that the reference to the problems' context was abandoned, in particular through the generation of expressions where the minus sign could not be given an immediate *non-algebraic* interpretation, but also through a process of manipulation of expressions that could only be meaningful in the context of the algebraic method of solution (not enabling, as we said before, an immediate *evaluation*). The ***internalism*** of those solutions imply their *arithmeticity*, and as it is reasonable to expect that most of those students would not justify their manipulation of equations on the basis of properties of numbers, this *arithmeticity* means instead a focus of attention on the arithmetical operations as a source of information on what could and should be done to solve the equations, thus the problems.

Much more frequently than not, algebraic solutions were *method-driven*, with the overall control and meaning of the process being related to the process of producing transformations leading to the special form

$$x = f(data)$$

while non-algebraic solutions were frequently constituted of a sequence of models, each one produced through the evaluation of a part or partial whole and manipulated locally, which in many cases led the students to disregard initial conditions or to introduce new ones. This is not, however, a necessary characteristic of non-algebraic models.

The relevant aspect we could detect in relation to the effect of teaching, is the greater flexibility of AH7 when compared to AH8. The younger AH7 group used mainly non-algebraic approaches where the problems were amenable to them, but were inclined to switch to an algebraic approach whenever they were not, **even when they did not have the necessary technique to deal with the resulting algebraic model readily available**. This effect had already been detected in the previous two sections, but the greater complexity of the questions in this group made it even more clear.

61 Loosing track of the variables means not being able to correctly associate the result of a series of *evaluations* with the parts or partial wholes it corresponds to; loosing track of the process of solution means disregarding one or more of the initial conditions of the problem.

4.5 The Buckets-Secret Number Problems

The Problems

From a tank filled with 745 litres of water, 17 buckets of water were taken.
Now there are only 626 litres of water in the tank.

How many litres does a bucket hold?
(Explain how you solved the problem and why you did it that way)

Buckets

Question 1

I am thinking of a "secret" number.
I will only tell you that ...

181 - (12 x secret no.) = 97

The question is: Which is my secret number?
(Explain how you solved the problem and why you did it that way)

Sec+

I am thinking of a secret number.
I will only tell you that

120 - (13 x secret no.) =315

The question is: Which is my secret number?
(Explain how you solved the problem and why you did it that way)

Sec-

General Description

The problems in this group were designed mainly in order to check the extent to which a *whole-part* model — the most natural model to use with the **Buckets** problem —

would be used to *model back* **Sec+** and **Sec-**. We expected **Buckets** to be easier than both **Sec+** and **Sec-**, and **Sec+** to be easier than **Sec-**.

The complexity of the problems was kept low in order that issues relating to the choice of model could be highlighted.

DISCUSSION OF POSSIBLE SOLUTIONS

All three problems could be modelled algebraically either directly, with an equation like

$$745 - 17x = 626 \qquad \text{(I)}$$

or first producing a reformation of the problem's situation to produce an equation like

$$17x + 626 = 745 \qquad \text{(II)}$$

corresponding in **Buckets** to the fact that the water taken, together with the water that was left, corresponded to the initial amount of water, and then solving it algebraically. Nevertheless, setting the equation could serve only to make the problem's statement more compact, with the solution proceeding from there non-algebraically.

Non-algebraic solutions for **Buckets** and **Sec+** would probably involve the same model, relying on the perception of a *whole-part* relationship, namely the one leading to equation (II), and solved on the basis that if one removes from the *whole* the *part* that remained, what is left is the *part* that was taken, and this resulting *part* would be *shared* between the 17 buckets or into 13 *parts*. In relation to **Buckets**, the procedure is very much analogical and requires no further modelling or interpretation; in relation to **Sec+**, there has to be an interpretation of the subtraction as "removal" and from there the *whole-part* relationship is established.

This model, however, is obviously inadequate to **Sec-**, and because it is impossible to avoid the acceptance of negative numbers at some point, this problem is naturally closer to the *Semantical Field of numbers and arithmetical operations*. This inadequacy accounts, in fact, for much of the importance of this group of problems in relation to the whole set of test problems; the low level of complexity allows us to better examine the effect of the "push" towards the *Semantical Field of numbers and arithmetical operations*. The two subtraction items involving negative numbers (**25-37** and **20-(-10)**) were designed to provide supporting information to the analysis of the responses to these problems and those in the group analysed on section 4.6, one of which also involves a negative number as the answer.

As we expected, a clear hierarchy emerged, with **Buckets** being the easiest problem, then **Sec+**, and **Sec-** being the most difficult. The differences in the facility levels were significant in all cases but between **Buckets** and **Sec+** in AH8 and in FM2, a fact that we will closer examine ahead. AH8 was the only group where the level of facility for **Sec-** was high (71%, against 14%, 15% and 17% for AH7, FM2 and FM3 respectively), and it is very significant that all those correct answers in AH8 were produced by solving an equation. As with all the previous problems we have analysed, the level of use of equations by FM2 and FM3 was very low.

The flexibility in the choice of approach previously shown by AH7 is also present here in a very clear manner. Although the facility level falls from **Buckets** to **Sec+**, the huge fall in the number of **OKCALC** solutions is compensated by an increase in the number of **OKEQT** solutions; moreover, on **Sec+** two-thirds of the incorrect answers are **WCALC**, but on **Sec-** this situation is more than reversed, with three-fourths of the incorrect answers being **WEQT**, and this is a good indication of their willingness to switch to an algebraic model when the non-algebraic models are not enabling them to solve the problem. AH8 also show some flexibility here, with almost two-thirds of their correct answers to **Buckets** being **OKCALC** solutions. On the **Sec** problems however, **all** their correct and incorrect solutions use an equation; the use of an algebraic approach is certainly responsible for the high level of facility for **Sec-** in AH8, indicating that in the case of this problem it represents indeed a more powerful tool for solving it than non-algebraic approaches. This will be examined more closely on the students' solutions.

Because **Buckets** and **Sec+** have an identical *whole-part* structure, the difference in the facility levels strongly suggests that many students could not interpret the arithmetical subtraction as a *removal* to produce a situation similar to the one in **Buckets**. Given that many students correctly used in those and previous problems a subtraction to *evaluate* the result of a *removal*, a subordination of the use of the arithmetical operation to the perception of the a *whole-part* model is established in this case, as opposed to some form of more or less symmetrical correspondence between *subtraction* and *removal*.

STUDENTS' SOLUTIONS

The Buckets problem

By far, the typical correct solution to this problem was an **OKCALC** solution. In most of those (38 out of 59 **OKCALC** instances) some explanation was given, making reference to the fact that to know how much had been taken on the buckets one had to subtract what was left from the initial amount of water (eg,Fabiana M, AH7; Sidnei A, AH7; Alexander P, FM2; Rebecca H, FM3).

$$\begin{array}{r} 745 \\ -626 \\ \hline 119 \end{array} \qquad 119 \;\underline{|\,17\;} \quad 00 \;\; 7$$

Pensei ... se tinha 745 e agora tem 626, quer dizer que tiraram
119 l de água em 17 baldes.
l ÷ quantidade = l.

Fabiana M, AH7

"I thought... if there were 745 and now there are 626, it means that 119 l. of water were taken on 17 buckets."

$$\begin{array}{r} 745 \\ 626 \\ \hline 119 \\ 7 \end{array} \qquad 119 \;\underline{|\,17\;} \quad 00 \;\; 7 \;\; 7$$

Fiz essa conta p/ saber quantos litros sai da caixa d'água

Eu usei essa conta porque se saiu 119 litros no total o lógico que tenho que dividir p/ saber quantos litros entram em cada balde.

Sidnei A, AH7

"I did this sum to know how many litres were taken from the tank. [at the left of script]

I did this sum because if 119 litres were taken altogether [,] the *logical* thing [is] that one would have to divide to know how many litres go into each bucket."

To find how many litres does a bucket hold we first have to find how many litres were taken from the tank
745 - 626 = 119 litres
119 litres were taken. There were 17 buckets. So ~~E~~ each bucket hold $\frac{119}{17}$ litres of water. The answer is 7 l.

Alexander P, FM2

I took 626 away from 745 which left me with 119 divide 119 by 17 and got 7. I did this becouse if you take 626 away from 745 you get the amount of water taken then divide by 17 because there are 17 buckets.

Rebecca H, FM3

Sidnei's reference to "the *logical* thing to do" seems to be his way of saying that no explanation is necessary as to why it is so. In all four scripts the subtraction part of the procedure is taken as self-evident; in no case an explanation is provided as to why this subtraction correctly provides the amount taken, not in verbal terms nor using some kind of diagram. Also, in none of the solutions the intermediate step of saying or showing that the amount taken plus the amount left corresponded to the initial total amount was taken. Altogether, this is an exceptionally strong indication that the direct procedure was perceived as an intrinsic property of the situation and the explanation would only have to indicate which numbers corresponded to which "roles." Similarly, no explanation was ever provided as to why the division by 17 produced the amount taken on each bucket.

Only six solutions used equations, five correctly solved and one incorrectly solved. Flávia C (AH7)[62] first makes a mistake by writing 75 instead of 745 on the initial equation; then, instead of the correct — in that context — 75-626 subtraction, she does 626-75. This "corrective" manipulation probably corresponded to the perceived need to produce a positive number as the answer or to a pre-equation perception of the calculations required to solve the problem. The latter seems to be a better interpretation, as hers is the only of the six solutions using the equation

[62]The text on Flavia's script simply explains that "17x...means...17 times x."

$$a-bx=c$$

where the first step of the solution leads to

$$bx=d$$

and not to

$$-bx=d$$

strongly suggesting that her solution uses algebraic notation but is guided by a *whole-part* model as in the **OKCALC** solutions examined above, and the 626-75 subtraction simply corresponds to "initial total minus remaining water", where the smaller of the two numbers obviously had to play the role of "remaining water".

$75-(17.x)=626$

$17x = 551$

$x = \frac{551}{17}$

$x = 32,4$

Você fazendo 17x, quer dizer que foram tirados 17 vezes x de água de um tanque de 745 litros que foi para 626 litros.

R: Cabem 32,4 litros em cada balde

Flávia C, AH7

In only one of those six solutions using equations, Andrea T's (AH8), the initial equation does not correspond literally to the problem's statement, corresponding instead to the statement "the water in the buckets together with the water that remained is the water one had originally" — obviously derived from the problem's statement.

$17x + 626 = 745$

$17x = 119$

$x = 7$

R: cabem 7 litros em cada balde

explicação - eu somei 17 baldes multiplicado por x, porque eu não sei a quantidade de água que contém em cada balde, com a água que restou e dá como resultado a água que tinha antes

Andrea T, AH8

"explanation- I added the 17 buckets multiplied by x, because I don't know the amount of water in each bucket, with the water that was left, and [I] gave as the result the water that was there before."

Andrea's procedure displays a characteristic similar to the *direct non-algebraic substitution* procedure we examined in relation to the problems in the Choc-Carp group of problems, by manipulating a non-algebraic model first, and then producing an equation from there. All other four **OKEQT** solutions proceeded without going through the equation

$$17x + 626 = 745$$

preferring instead to operate directly with the negative coefficient of **x** (eg, Ana RW, AH8). In Andrea's script we also find a clear example of the *analiticity* and *arithmeticity* of algebraic solutions.

$745 - 17x = 626$ $\quad$ $745 - 17(7) = 626$
$-17x = 626 - 745$ $\quad$ $745 - 119 = 626$
$(-1)\ -17x = -119$
$\boxed{x = 7}$

Cabem 7 litros em cada balde.
Encontrei o valor através de uma equação.

Ana RW, AH8

The seven **WCALC** solutions do not provide any interesting insight or instance.

The Sec+ problem

Characteristic of the **OKEQT** solutions is that here — as before with the **OKEQT** solutions for **Buckets** — in all cases but one the equation initially set corresponds directly to the problem's statement. Also — and more important, given that the problem's statement directly suggest a specific equation — in all instances, the solvers accepted and dealt with a negative coefficient for **x**, rather than first producing the transformation into

$$181 = 12x + 97$$

In two **OKEQT** scripts are displayed peculiar aspects of thinking algebraically. First, in Fabio C's (AH7) solution, one sees the constitution of a new *object* (**12x**), meaning more than a syncopated notation for the multiplication — even if slightly more; in his solution Fabio *operates arithmetically with the unknown.*

número $181 - (12 . x) = 97$ $181 - 12x = 97$ $-12x = 97 - 181$ $-12x = -84$ $x = 7$	Eu primeiro resolvi a operação que estava entre parênteses $(12 . x = 12x)$ depois eu resolvi o resto do problema como se fosse uma equação normal.

Fabio C, AH7

"First I solved the operation in brackets (12 . x = 12x) then I solved the rest of the problem as if it were an ordinary equation."

Christiane T's (AH8) script is a fine example of the *method-driven* aspect of algebraic solutions, as she multiplies the second equation by -1 even before performing the calculation on the right-hand side of the equation, in a sense treating the known numbers as unknown ones, but actually showing the extent to which the distinction between known and unknown numbers has faded.

$181 - 12x = 97$ $(-1)\ -12x = -181 + 97$ $12x = 181 - 97$ $12x = 84$ $x = \frac{84}{12}$	$x = 7$	Eu resolvi com a forma de equação.

Christiane T, AH8

In three scripts algebraic notation is used but the solution process is not algebraic. Célia R (AH7) solves the problem by first restructuring into the equivalent of "the amount that was taken corresponds to the difference between the initial and final amounts"; from there she writes and solves an equation, and one cannot positively decide whether there was a shift into the *Numerical Semantical Field* or whether she was using literal notation to describe a non-algebraic solution. In any case, the main step that allows her to *evaluate* **x** — the manipulation that led to the first equation — was most likely based on the perception of the *whole-part* relationship. In the other script the situation is much more clear, as Marcelle D (AH7) writes down the equation directly derived from the problem's statement, but the rest of the solution is void of further use of literal notation, and the solution process corresponds directly to one guided by the *whole-part* relationship. Finally, Gil S (AH7) uses literal notation only to express the general form of the procedure he used, possibly as a way of justifying it; we think that on the light of what we have said so far, there should be little doubt that his solution was guided by a *whole-part* relationship.

Conta | $181 - (12 \times 7) = 97$
$181 - 84 = 97$

$12x = 181 - 97$
$12x = 84$
$x = 84 : 12$
$x = 7$

Célia R, AH7

nº secreto = x
$181 - 12 \cdot x = 97$
$181 - 97 = 84$
$84 \div 12 = 7$, resto 0
$181 - 84 = 97$

Marcelle D, AH7

181 = Nº
Nº secr = X
97 = R
12 x Nº sec = S
Nº − (12 · x) = R
Nº − R = S ; 12 = X
$181 - 97 = 084$; $84 \div 12 = 7$, resto 0
O NÚMERO SECRETO É 7

Gil S, AH7

In most of the **OKCALC** solutions the explanation provided indicates that the *whole-part* relationship was on the basis of the solution process (Simon J, FM3; Sarah G, FM3; Marcelo A, AH7; Leandro F, AH7; Jennifer J, FM3).

The secret number is 7
I took away 97 from 181 = 84. This leaves me with the amount of A, A is 12 × something so if i divide 84 by 12, it will give me the number

Simon J, FM3

$181 - 97 = 84$

$12 \times ? = 84$

$12 \times (7) = 84$

$181 - (12 \times 7) = 97$

Sarah G, FM3;

181
−97
084 | 12
7 número secreto

Eu subtraí 97 de 181 para saber qual era o outro fator, e posteriormente dividi este número por 12 para descobri o número secreto.

Marcelo A, AH7

"I subtracted 97 from 181 to know which was the other factor..."

181 − 97 = 84 | 12
00 7

Eu resolvi porque se o resultado = 97 então 181 − 97 vai dar o resultado [illegible] e então dividimos por 12 e da [illegible]

Leandro F, AH7

"I solved [it like this] because if the result=97 then 181-97 will give the result of the multiplication..."

181 − 97 = 84 thats the sum for the secret number. 84 ÷ 12 = 7 so that must be the secret number.

Jennifer J, FM3

It is central that the form in which it is expressed is of no importance, as the *decomposition* process is always clearly visible. The use of a letter (the "A" in Simon's script), a verbal specialised term ("factor", in Marcelo's), or a more or less standard, non-literal notation (the question mark in Sarah's) do not make the solution essentially distinct from those using verbal, relatively neutral references ("the multiplication" in Leandro's or "the sum" in Jennifer's). As with the **OKCALC** solutions to **Buckets**, there was never any explanation as to why the subtraction would produce the remaining the value of the remaining *part.*

In some of the incorrect solutions the source of the errors can be traced back to the use of loose and incorrectly generalised, verbally formulated rules like "undo it using the inverse operation" or the rules for the manipulation of algebraic expressions (Rebecca H, FM3; Sukhpal S, FM3; Ana Lúcia E, AH7). Nevertheless, in this kind of behaviour one can identify the focus of attention being at the arithmetical operations — even if it does not result in correct procedures — and this evidences at least a willingness to limit one's attention to the arithmetical context, a necessary aspect if one is to operate within the *Semantical Field of numbers and arithmetical operations.*

Secret no

I took the 97 and + it by 181 because it is the opposite of ~~-~~ and with that number di ÷ ~~[illegible]~~ 12 because it is the opposite of ×.

Rebecca H, FM3

$\cancel{\$}181 - (12 \times x) = 97$

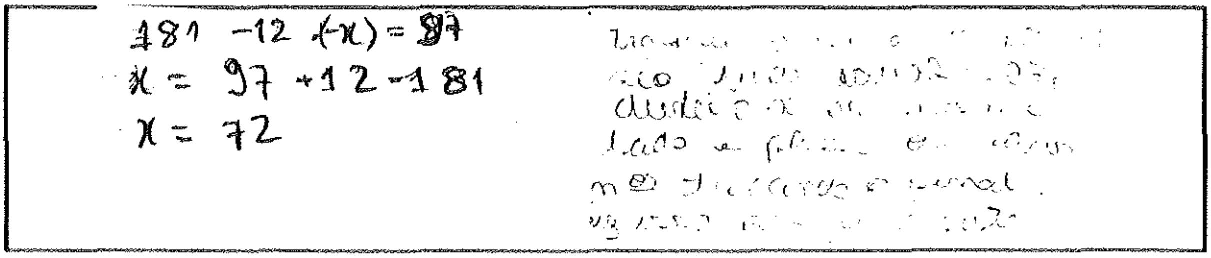
$= 97 \div 12 - 18 = x$

$= 12\overline{)97}\ 8\,r1 \quad + 181 = 2\cancel{0}71$

$= x = 26r1$

$=$ secret No. $= 2\cancel{6}71$

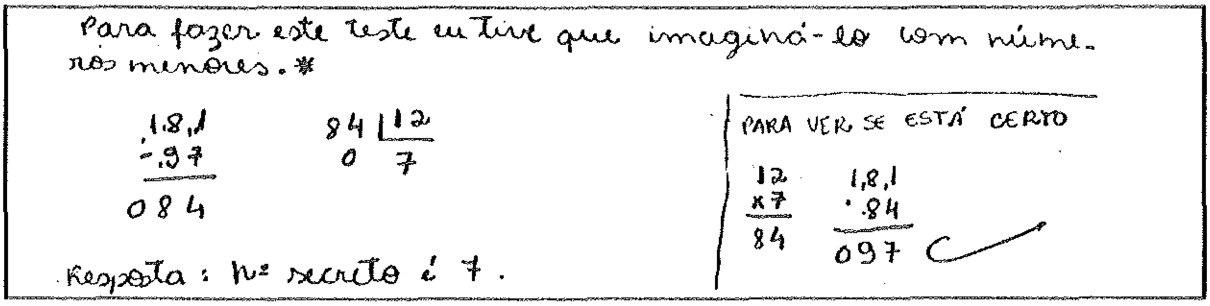
Sukhpal S, FM3



Ana Lúcia E, AH7

"I changed the sign of the parenthesis..." [as if it were an addition or subtraction instead of a multiplication]

One script in this group is of interest to us, because it employs a unique approach (Cecília B, AH7).



Cecília B, AH7 (solution to Sec+)

"To do this test I had to imagine it with smaller numbers"

on the left, parallel to the margin: **36 - (2 · secret no.) = 20**

on the right-hand corner: "to see if it's correct"

From the simpler example, Cecília works out the string a calculations that leads to the solution of the equation, and simply applies it to the original numbers. On one hand, her solution seems to rely completely on insights emerging from the simpler example; the solution is thoroughly *synthetical*. On the other hand, she easily accepts that the "algorithm" can be applied to a problem from which she did not feel able to derive the solving steps, ie, that the range of numbers to which it applies is not dependent on properties of the small numbers on the "exemplary" case and the relevant factor is the *numerical-arithmetical* structure. Even more striking, Cecília applies exactly the same method to solve **Sec-** (script also shown bellow), and the "simpler problem" she uses with **Sec-** is not, as one might have expected, in direct correspondence with its statement, where the "result" (ie, the number on the left-hand side) is greater then the "starting number" (ie, the number from which a multiple of the secret number is subtracted). The "simpler problem" she invents is

$$20 - (4 \times \text{secret no.}) = 12$$

from which, knowing that the secret number is 2, she correctly derives the solving algorithm as

$$\text{secret no.} = \frac{20-12}{4}$$

The crucial step, thus, is that she puts in correspondence the numbers in this model with the numbers in the problem's statement, *regardless of the fact that in Sec- the "result" is greater than the "starting number,"* and correctly applies the algorithm, paying attention to the order of the terms in the subtraction and of the sign of the final answer. It is clear that the process is carried out completely within the *Semantical Field of numbers and arithmetical operations*, as control of the operations depends totally on the *arithmetical articulation* of the paradigmatic expression. Hers, however, is not an *algebraic* solution, as it is *synthetical* by the very nature of the solving technique.

Para descobrir, eu inventei este outro problema:
20 - (4 x nº secreto) = 12. Sei que o nº é 2. Então
eu vi como se pode, com esses números chegar em 2.
ENTÃO:
120 - 315 = -195
-195 : 13 = -15
Resposta: O nº secreto é -15.

Cecília B, AH7 (solution to Sec-)

"To find out, I invented this other problem:

20- (4 x secret no) =12. I know that the secret number is 2. So I saw how one can, with those numbers, to get to 2.]

Then..."

Finally, we have Melissa R (FM3). The first step of her solution — *evaluating* "what is between the brackets" — seems clearly based on the *whole-part* relationship. The second step, however, instead of representing an *evaluation* of the *sharing* is explicitly a manipulation of the newly established relationship, namely **$12 \cdot x = 84$**, based on syntactical transformation. We would not go so far as to say that she was fully aware that the "reversing of the multiplication **sign**" stands in fact for a property of the operation, but the source of information on what to do next was certainly the *numerical-arithmetical* expression, in particular the multiplication operation. We have thus a mixed solution. When she solves **Sec-** (script also shown bellow, together with the script for **Sec+**), she first concludes for the answer being 15 and only then adjusts the answer to -15 in order for it to fit the problem's statement (15 is encircled at the top-left corner of the script, and the minus sign at the end of the string of calculations on the first line was certainly inserted afterwards, looking "squeezed" between the equal sign and the number); the adjustment is made by assuming that the 195 had to be negative (and she puts a minus sign to the left of 195 on the first line, which is later obliterated). Her solution does not proceed through successive transformation of equations, but much of it is clearly performed within the *Semantical Field of numbers and arithmetical operations*; again, Melissa shows flexibility in mixing different models, but she is successful only due to the extreme care taken in seeing that the overall result was adequate in relation to the original condition set on the problem's statement.

The secret no. is 7. Take 94 from 151 because you want to get a no to help you with what is between the brackets. The no you get is 84. You then reverse the multiplication sign to a division and divide 84 by 12 to get the answer

Melissa R, FM3 (solution to Sec+)

15 $\quad 315 - 120 = 195 \div 13 = {}^{-}15$

Do this first to get the multiplication number

reverse the multiplacation sign and divide by 13 to get the secret number.

$13 \times {}^{-}15 = {}^{-}195 \quad 120 - {}^{-}195 = 315!$

Melissa R, FM3 (solution to Sec-)

The Sec- problem

The main difficulty in dealing with this problem using non-algebraic models is that the *whole-part* model that worked so smoothly with **Buckets** and **Sec+** simply does not make sense in this case, as Daniel S (FM2) puts it.

This is impossible because 315 is already higher that 120, so you can't take 13 x secret no. because it will end up even smaller But it could be a minus number.

Daniel S, FM2

The observation at the bottom line might indeed serve as the seed of a corrective approach that can be used to make a *whole-part* useful. By assuming the secret number to be negative, one immediately has that the subtraction notationally indicated is not "in fact" a subtraction, but an addition, and the problem is reduced to

120 + (13 x secret no) = 315 (equation I)

which can be easily solved with the help of a *whole-part* model. In Mi P's (FM3) solution the minus sign is added to the answer only after the "amount" is found; Sophie W (FM3) on the other hand, worked out the value of **13xsecret no** to be **-195** and proceeded from there by dividing it by **13**, as also did Jennifer J (FM3, script not shown). In both Mi P's and Sophie's solutions the main step relies on a property of numbers, but the use of the *whole-part* relationship is also crucial. The perception that the secret number is negative expresses not only the *numerical* treatment of the problem, but also some degree of *analiticity* in the approach, as the secret number — yet unknown — is taken as having a property, which means it has been made into an *object.*

120 − (13 x −15) = 315
I thought the number had to be a minus so it could equal 315.
All I had to do was find the difference between 315 − 120 = 195.
Then I had to find 13 times whatever the number was to equal 195 (15)

Mi P, FM3

~~315~~
~~−120~~
~~195~~

~~435~~
~~13⟌195~~

−195

13⟌195 = 15

~~13x15 = 195~~
~~195~~
~~+120~~

13 x −15

−15

Secret number

I worked out that the secret no needed to be a negative number then I worked out that 13 x secret no had to equal −195 so I divide −195 by 13.

Sophie W, FM3

Attempts to use a *whole-part* model lacking the perception that the secret number is negative, led to two types of error. In eight cases the solver simply assumed that **315** corresponds to the *whole* and that **120** and **13·secret no** correspond to the *parts* (eg, Marcelo A, AH7), as if the problem said

315 - (13 x secret no) =120

and the problem is solved as **Sec+** would be using a *whole-part* model.

315
-120
195

195 | 13
65 | 15

Primeiro eu subtraí 120 de 315 para saber qual era o número dos parênteses e depois dividi este número por treze.

número secreto

Marcelo A, AH7

"First I subtracted 120 from 315 to know which was the number in the brackets and then divided this number by thirteen."

He encircles 15 and writes "secret number"

We can safely conclude that this inversion is caused by the "meaninglessness" of the original statement in terms of wholes and parts, as expressed by Daniel S two paragraphs above, representing an attempt to make sense of the situation, as **all** eight student who produced this type of solution had solved **Sec+** using a *whole-part* model. Another inversion produced by students in the problem's statement was to take the subtraction

120 - (13 x secret no)

as actually indicating

(13 x secret no) - 120

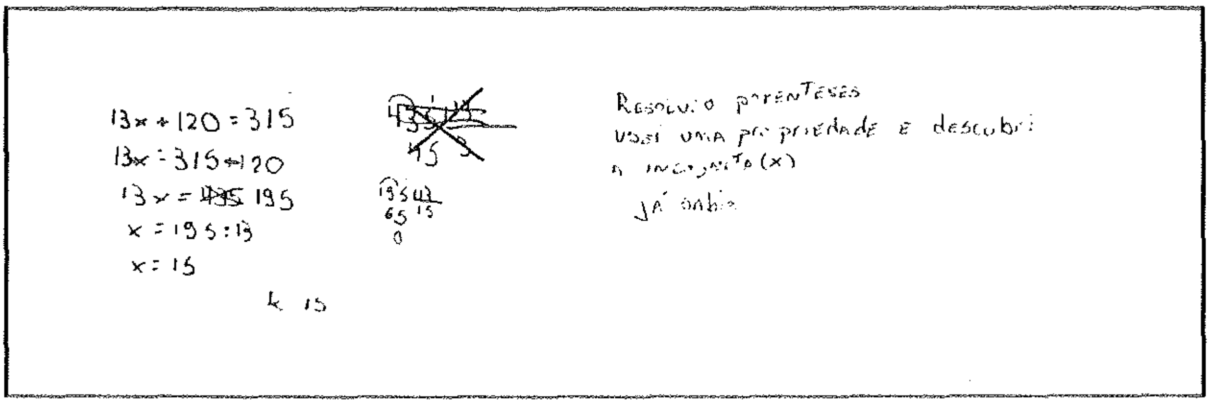
which also restores the meaning in terms of wholes and parts (David B, FM3).

315 + 120 = 435
435 ÷ 13 = (secret number) 33 R6
I added 120 onto 315 because in the sum it was to be taken away. Then I divided it by 13 as the secret number was timesed by 13.

David B, FM3

Five students produced this type of solution; only two of them had correctly solved **Sec+**, both **OKCALC** solutions, one was a **T&E** solution, one was **NATT**, and in one case a similar error was made there as here. If one thinks in terms of a hierarchy, it seems that incorrectly reversing the terms of the subtraction (second type of error) represents a cruder error than adjusting the roles of the numbers involved (first type of error), as the students doing the latter error seemed to be operating much closer to a consistent model for dealing with problems of this kind[63].

To one of the students, Luís N (AH7), the drive to make sense of the problem's statement in the context of whole and parts was so strong that he simply "corrects" the statement, to produce equation I we showed a few paragraphs above, without realizing that the number coming from the new equation would have to be adjusted to fit the problem's condition[64].



Luís N, AH7

"I solved the brackets

used a property and found out the unknown (x)

I already knew it [how to do it]"

Marcelle D (AH7) uses algebraic notation; at first sight it might seem as if she simply misapplied rules for the manipulation of equations[65]. On the light of the analysis of the previous few paragraphs, however, we are led to conclude that in fact she made sense of the equation by producing the same reversion of the subtraction as David B above. Her solution to **Sec+** also begins with an equation, but proceeds with calculations only.

[63]Disregarding the order of the terms in a simple subtraction is a mistake that has been identified by several researchers, and it might have contributed to making the mistaken reversing more acceptable to those students.

[64]It is impossible to decide from the script only whether he solved the resulting equation by thinking algebraically or whether he stayed with the whole-part model, but because of the seemingly cause for the "correction", we would — more as a matter of exercising interpretation than as a matter of this decision being relevant — prefer the latter interpretation.

[65]Namely, "change sides, change sign", with the "-" sign seen as "belonging" to **120**.

$$120-13x=315$$
$$13x=315+120$$
$$13x=435$$
$$13x=435\div 13$$
$$x=33$$

Marcelle D, AH7

As it happened on **Sec+**, almost all **OKEQT** solutions reached at some point the equations

$$\mathbf{-13x = 315 - 120} \text{ or } \mathbf{-13x = 195}\ ;$$

in most of them the solver multiplied both sides by -1 (Flávia C, AH8) to obtain

$$\mathbf{13x = -195}$$

and in a few cases the solver carried on with **-x**, dividing first by 13 and only at the end reversing the signs on both sides. Fábio C (AH7) directly reaches an equation of the form **13x=...**, but this step is justified in terms of the process of solving the equation, and not in terms of a relationship derived from the initially given *whole-part* relationship. It is significant that this form of control of the process results in a correct derivation, while Marcelle — even with the support of literal notation — and other students whose solutions were guided by a *whole-part* model failed. By shifting the *meaning* of the process into one closely related to the *method of manipulation of the expressions*, away from the context of *evaluation of measure of parts*, Fábio's approach overcomes the difficulties involved in making sense of this problem within a *whole-part* semantic.

$$120-(13\cdot x)=315$$
$$120-13x=315$$
$$-13x=315-120$$
$$-13x=195$$
$$(-1)\ 13x=-195$$
$$x=-\frac{195}{13}$$
$$\boxed{x=-15}$$

R.: Eu substitui o número secreto pelo "x" e resolvi a equação de 1º grau.

Flávia C, AH8

$$120-(13\times \overset{\text{(nº secreto)}}{x})=315$$
$$120-13x=315$$
$$120-315=13x$$
$$-195=13x$$
$$x=\frac{-195}{13}$$
Resposta → $\boxed{x=-15}$

Eu resolvi como se fosse uma equação.
Eu, primeiro resolvi o parentese; depois eu passei o nº secreto (x) para um lado e os números para o outro, aí é só resolver a equação.

Fábio C, AH7

"I solved as if it were an equation.
First I solved the brackets, then I moved the secret number (x) to one side and the numbers to the other, then it's only to solve the equation."

In several **WEQT** solutions, the solver arrives at either

$$\mathbf{-13x = 195} \quad \text{or} \quad \mathbf{13x = -195}$$

only to produce **15** (instead of **-15**) as the answer. Difficulties with the division involving a negative number could certainly be responsible for the incorrect result, but one script suggests another possible source for it (Ana C, AH8). Although keeping the algebraic correctness at a syntactical level — in this case, keeping the coefficient of **x** negative — it is possible that the model underlying the reasoning was in fact based on the perception of a *whole-part* relationship; in Ana's script this is indicated by the fact that she refers to "the number 'x'" — probably a reference to the amount taken — and also to it being "'13x'", but she never refers to the negative coefficient or to the fact that her reasoning would have to be complemented by something like "but in fact each **x** is negative". The perception that the result had to be a negative number did not come from the awareness that "I subtracted something and it got bigger" nor from the recognition that the coefficient was in fact **-13** and not **13** — and thus the divisor would have to be **-13** were she "reversing" the multiplication. Both aspects being essentially *numerical-arithmetical*, this lack of understanding supports the case that the model underlying her solution process was indeed a non-algebraic one. Ana's solution to **Sec+** (script bellow) is similar in this respect to the solution to **Sec-**, as she correctly keeps the minus sign but does not deal directly with it (when most **OKEQT** solutions did), and the process produces a correct result only by virtue of the "friendliness" of the problem; the written explanation certainly corresponds to a solution guided by a *whole-part* model[66].

[66]One might argue that she justifies the division as reversing the multiplication and this brings the solution closer to an algebraic one, but we think the crucial and characteristic step here is deriving **12x=84** from the initial statement, as in algebraic terms this would involve directly manipulating the unknown.

$-13x = 315 - 120$

$-13x = 295$

$x = \frac{295}{13}$

Se você subtrai 315 de 120 você [illegible]

Ana C, AH8 (solution to Sec-)

"If you subtract 315 from 120 [sic] you'll have the number "x". But as there "13x", you have to divide by 13."

$181 - 12x = 97$

$-12x = 97 - 181$

$12x = 84$

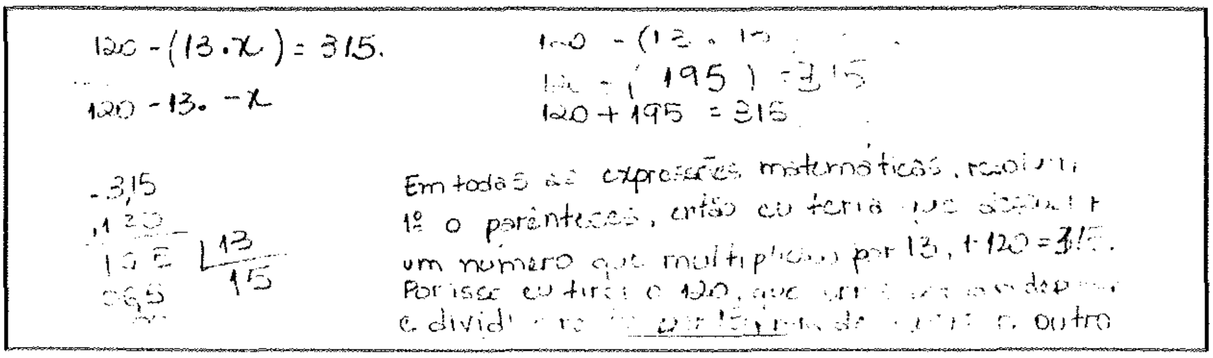
$x = \frac{84}{12} = 7$

Você tem um número (181) que tirado ao outro número que não se conhece dá um resultado (97). Se você tirar a quantia do resultado (97) da 1ª quantia você terá a diferença entre os 2. Como 12 está multiplicando você o passa p/ o outro lado dividindo.

Ana C, AH8 (solution to Sec+)

"You have a number (181) that taken <u>from</u> the unknown number [our emphasis] gives a result (97). If you take the amount of the result (97) from the 1st amount, you'll have the difference between the two. As 12 is multiplying, you move it to the other side dividing."

Fabiana M's (AH7) script is very interesting for several reasons. At first she tries setting and solving an equation, and it seems that she tries to "distribute" the minus sign over **13x** (top-left corner); as the resulting expression is not *meaningful* to her, ie, she cannot get information on how to proceed with the solution from it, she shifts to another model, which is clearly based on a perceived *whole-part* relationship. From the verbal explanation we learn that she had already transformed the problem — inadequately — into one equivalent to the additive equation I some paragraphs above ("...a number that multiplied by 13, +[!]120=315..."). We think it is extremely significant that the model takes control of the solution process to the extent that the simple arithmetic rules are subordinated to its *semantic*; it is enough to observe that on the three lines of expressions (top, center-right), the subtraction notationally indicated is never meant to be one, as it is revealed on the third line. Fabiana had solved the item **25-37** correctly, which indicates that the disregard for the rules of arithmetic were not a mistake but part of *operating in another Semantic Field.*



Fabiana M, AH7

"In all mathematical expressions we first solve the brackets, then I would have to find out a number that multiplied by 13, +120=315. That's why I took the 120, that would be added later, and divided the rest by 13 to find out the other number."

Leandro F's (AH7) solution offers us a rare instance of algebraic thinking without manipulation of literal notation or algebraic expressions. The expression he derives for the secret number is correct, and it takes into account that if the secret number is to have a positive coefficient — or, as he would possibly put it, "for the secret number to be 'positive'" — the correct subtraction is **120-315**, and he also uses the brackets correctly. We think Leandro's solution is substantially different from those in which an awareness that the secret number was negative existed but the solution process proceeded within the context of the additive equation, and this difference is clearly shown by the fact that from the beginning the terms involved in the calculations he indicates are correctly signed; there is no transformation of the problem with an adjustment *a posteriori* to fit the original condition of the problem. His verbal explanation is very confuse, and almost nothing more can be gathered from it; we produced a very literal, almost word-by-word translation in order to convey this state of things. For all we said above, the fact that his final answer is **15** and not the correct **-15** only supports our interpretation, once it indicates that he was not aware beforehand that the answer had to be negative, and produced the necessary transformations on the basis of his perception of the numerical structure of the problem's statement.

nº secreto = (315 + 120) ÷ 13
nº secreto = − 195 ÷ 13
nº secreto = 15

195 | 13   315
65   15   − 120
00   195

Eu descobrir que era menos por causa do sinal de – na frente do parenteses, e também deu para [illegible] que o resultado -120 e quando eu fiz a conta e dividi ao treze para ver se dava

Leandro F, AH7

"I found out it was minus because of the - sign in front of the brackets and also it was possible to know that the result-120 and when I did the calculation and divided by thirteen to see if it would be possible."

Finally we examine Vicky H's (FM3) script. There are two points of interest. First she rewrites the problem's statement using letters not only for the unknown, but also for known numbers. According to our traditional usage, she is not distinguishing the known numbers substituted from the unknown one, as the choice of letters seems to indicate a mere sequential A-B-C from left to right. On the other hand, she distinguishes A and C as having a different role than **13**, which she left as a definite number. We think that she was trying to put the problem's statement in a general form from which she could derive a pattern and a solution procedure. The generalised form she attained appears to bring three things into consideration:

(i) a possible *whole-part* model, which does not fit back into the problem's statement, as C<A (and she crosses out the generalised expression)

(ii) the perception that the subtraction had in fact to represent an increase, and thus an addition (and she concludes that "275 are needed"), and

(iii) the perception that the secret number had to be negative in order for the subtraction to result in an addition (and she gives as the answer -2.5).

There is no reference as to how she found those numbers, which are thoroughly incorrect. Nevertheless, her solution exemplifies the process of trying to make sense of the problem, and the successive changes in the understanding of the problem through this effort. The conflict between the general *whole-part* scheme and the situation posed by the problem is clear, as also are the necessary intervention of a knowledge of how numbers behave and the disadvantage of having to search through different new models when an algebraic model would be equally adequate for A>C and A<C.

$A = (13 \times B) = C$. 275 needed .
Secret number is $-2 \cdot 5$ (I think)

Vicky H, FM3

SUMMARY OF FINDINGS AND CONCLUSIONS

As we expected, a hierarchy appeared in relation to the facility levels of the three problems, with **Buckets** being the easiest and **Sec-** the most difficult; although the difference between **Buckets** and **Sec+** is not significant for AH8 and FM2, in AH8 there is a definite shift towards solutions using equations in **Sec+**.

Of all students, 83% correctly solved the item **25-37**, and 56% correctly solved the item **20-(-30)**, which strongly suggests that the inability to produce correct solutions to **Sec-** without using equations is due to the students' lack of willingness to operate *numerically*, ie, within the *Semantical Field of numbers and arithmetical operations*; this behaviour had been observed on the analysis of the previous groups of problems, but what makes it particularly significant here is the fact that **Sec+** and **Sec-** are not only identical in terms of their *arithmetical articulation*, but also all the one-step strategies that are available to reduce **Sec-** into a problem that can be modelled by a *whole-part* model — eg, presuming that the subtraction "is in fact" an addition", or simply considering the solution to **Sec+** and applying it as an algorithm to **Sec-** — depend in varying degrees on operating *numerically*, and the low level of complexity of the problems only highlights this aspect of the students' difficulties.

The percentages quoted at the beginning of the previous paragraph also accentuate the significance of the fact that many students reconstructed the problem in order to make it meaningful within the context of wholes and parts, showing that for many students the first-choice model is a non-algebraic one, in particular, a non-numerical one. Cecília's script establishes with great exactness that an analogy can be built between **Sec+** and **Sec-** in a way to engender a method to correctly solve **Sec-**, but this analogy is only possible within the *Semantical Field of numbers and arithmetical operations*.

Fabiana's solution, on the other hand, shows that the meaning of arithmetical operations can be adjusted to one's use according to the model being employed when one is operating in a *Non-numerical Semantical Field*. The important insight here is that many "mistakes" that have been used by researchers to characterise *misconceptions* might in fact be *conceptions within a Semantical Field other than the one intended by the researcher*, ie, it might be truly useful to consider that those students are not in fact thinking of what the researchers thought they were.

One important aspect related to the use of algebraic notation emerged. We had seen in solutions to previous "secret number" problems that employed equations that the substitution of specific symbols for "secret number" — usually **x** — was taken by many students as making the problems' statements into equations. In the explanations to their handling of **Sec+** and **Sec-**, a number of students referred to "**13x**" being the <u>result</u> of "**13·x**", revealing that the notion of *representation* was not readily available to them; this is a central part of *meaning* in *algebraic thinking*, and we think the lack of such understanding might represent a substantial obstacle in dealing, for example, with substitution solutions to sets of simultaneous equations. Also, the lack of the notion of *representation* might constitute an obstacle to the development of an understanding of thinking algebraically as proceeding within the *Numerical Semantical Field*, and thus, an obstacle to the constitution of the notion of *numerical-arithmetical structure.*

Finally, a few scripts—in particular Sophie's and Mi's—threw light into the use of algebraic and non-algebraic approaches on different stages of the same solution process, highlighting the possibility of usefully combining algebraic and non-algebraic models, and at the same time emphasising the dissimilarities between them.

4.6 PATTERN-SALESPERSON-SECRET PROBLEMS

THE PROBLEMS

Charles sells cars, and he is paid weekly.
He earns a fixed £185 per week, plus £35 for each car he sells.

This week he was paid a total of £360.

How many cars did Charles sell this week?
(Explain how you solved the problem and why you did it that way)

Salesperson

Her you have a pattern of tiles:

One possible formula that gives the number of white tiles that go with a certı number of black tiles is:

no. of whites = (2 x no. of blacks) + 6

How many black tiles are needed, if I want to use 988 white tiles?
(Explain how you solved the problem and why you did it that way)

Pattern

I am thinking of a "secret number".
I will only tell you that

(6 x secret no.) + 165 = 63

The question is: Which is my secret number?
(Explain how you solved the problem and why you did it that way)

Secret

GENERAL DESCRIPTION

(i) **Patt,** is a problem where both the generation of a pattern of black and white tiles and a formula relating the number of tiles of each colour on any composition respecting the pattern are given; the central objective was to investigate whether students would prefer to solve the problem reasoning directly from the spatial configuration or would use the formula given, and how they would manipulate those referents;

(ii) **Salesp,** is a very elementary problem about a salesperson who earns a fixed salary plus commission for each item sold; we never expected this problem to offer any difficulty to our students. It was included with the main objective of verifying how the students would justify the choice of arithmetical operations employed — would any justification at all be offered; we expected students to *explain* the use of the operations (eg, an addition used to know...) but not to *justify* the choice in terms of a more general scheme, numerical or otherwise, the reason for our expectation being the great familiarity with the type of situation[67]. The Brazilian version uses fridges instead of cars to make the numbers in the problem smaller.

(iii) A "secret number" problem, **Secret**, is stated in a syncopated form, rather than the usual verbal one; in this problem the solution is a negative number, and we expected it to be significantly more difficult than the other two. It was included in this group to allow us to examine the models produced in a situation where a *whole-part* model is not immediately available.

DISCUSSION OF POSSIBLE SOLUTIONS

All three problems in this group can be solved with an equation in one unknown,

$$\mathbf{b + ax = c}$$

If this approach is used, the three problems would present a very similar facility level, as the only one where an equation is not immediately given, **Salesp**, is very straightforward in verbal structure.

Patt offered the alternative of working on the basis of perceiving, for example, that if the three white tiles at each end of the pattern are removed, one is left with a simple 2 whites to 1 black ratio. From this point of view, the formula provided with the problem's statement would be an unfortunate choice, as the *non-algebraic* procedure we have just described would use the same calculations as algebraic solutions employing the formula, and this makes the more difficult to distinguish between approaches. However, the alternative would be to give, instead, a formula such as

$$\textbf{no. of whites} = 2 \times (\textbf{no. of blacks} + 2) + 2$$

which is obviously more complex than the one we decided to use, making a direct comparison with **Secret** — an important point — more difficult.

Secret could also be solved through the perception that the answer had to be a negative number, leading to the transformation of the problem into

[67] Another situation equally typical and familiar would be, for example, a problem involving change and the buying of several of the same items.

$$165 - 6n = 63$$

which would be solved as **Sec+** in the previous group of problems, possibly based on the whole-part relationship.

The obvious solution to **Salesp** would be to consider that the total income is composed by the fixed part together with the commission for sold items, so to know how much came from selling, it is only necessary to take the fixed part from the total income, a procedure based on the perception of a whole-part relationship.

GENERAL DATA ANALYSIS

Two unexpected results emerged. First, the overall facility level for **Secret** was 56%, much higher than we expected, specially if one considers that the other "secret number" problem with a negative answer (**Sec-**) had one of the lowest facility levels of all problems (27%)[68]. Second, in the Brazilian groups **Patt** was more difficult than **Secret,** while in the English groups this is not the case; this fact is surprising given that **Patt** offers not only the equation but also the support of a diagram, and even more so if one considers that AH8 proved to be very proficient in solving equations. One likely explanation is that the context of a pattern of tiles might have confused the Brazilian students, as this is a very unlikely context for a problem in Brazilian schools, while it is a very common one in English schools. A close examination of the students' solutions will provide further insight on the reasons for this result.

Also unexpected was the very low level of facility for **Patt** in FM2 (18%), as this problem should be familiar to them and offers no difficulty with the numbers. Nevertheless, while for **Secret** 71% of the scripts were **NATT**, 53% of the students in FM2 attempted a solution to **Patt** and failed, suggesting that they at least felt the possibility of producing a correct solution.

In agreement with the result of the previous groups of problems, the Brazilian groups preferred to use equations whenever they were suggested (**Pattern** and **Secret**), while in the English groups equations were used by only one student in **Secret**.

Salesp was the easiest problem in all four groups, with an overall facility level of 84%, identical to that of **Buckets**, in the last group of problems we analysed. As the

[68]Although the facility level in AH8 is very high (89%), forcing the overall result up, one has to observe that the percentages for AH7 and FM3 are very much in agreement with the overall result.

scripts will further demonstrate, those two problems were treated in very much the same way, with the choice of operations being taken as "logical" and never justified.

STUDENTS' SOLUTIONS

The Patt problem

All but one correct solutions to **Patt** from the English students — most of them on the third year group — were **OKCALC**, and many of them were justified by appeal to "reversing the formula", "reversing the procedure", etc..(Ian C, FM3; Joe V, FM3; Katy S, FM3).

Formula = ×2 +6 = Answer

988 ÷ 6 = 164.6

164.6 − 2 = 162.6

"incorrect answer."

" now I will reverse the formula"

988 − 6 = 982.

982 ÷ 2 = 491.

"You will need 491 black tiles."

Ian C, FM3

To work out the number of black tiles from 988 white tiles you have to reverse the formula so –

no of whites = (2 × no. of blacks) + 6 ~~becomes~~.

becomes

no of blacks = (no. of whites − 6) ÷ 2

notice that '+' becomes '−' and that '×' becomes '÷'.

Therefore the answer to the problem is

no of blacks = 988 − 6 = 982 982 ÷ 2 = $\boxed{491}$ the answer.

Joe V, FM3

To work out the number of black tiles, the formula needs to be reversed: e.g
Instead of x 2 + 6. It is -6 ÷ 2. I can check this on an example 14 whites - 6 ÷ 2 = 4 blacks. This is correct.
Sum: 988 wt - 6 ÷ 2 = 491 black tiles I can check this back on the example given w = 2x491 + 6 = 988, so it's right.
Answer: 491 black tiles

Katy S, FM3

Although this type of justification was given to other problems, what is remarkable here is the high proportion of students producing it, together with the specific notation used by some students, suggesting a strong influence of taught models. No student actually used a "boxes and arrows" diagram (figure Patt 1), but the treatment of ÷**2** and +**6** as operators, rather than treating **2** and **6** as operands, is clear. Those solutions are *numerical-arithmetical,* as they are guided by properties related to the arithmetical operations only (as it is made clear in Joe's solution), but they are not *analytical*; the secret number is perceived as an initial state and never directly manipulated. Also, the solution process concentrates only in producing "the way back", so to speak, and the transformation of the arithmetical operations into their inverses never involves the manipulation of a *numerical-arithmetical* relationship.

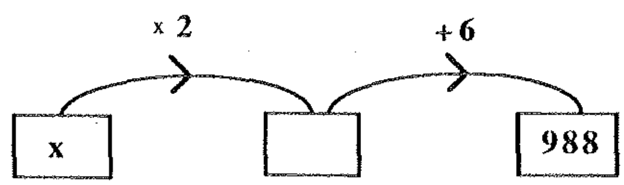


fig. Patt 1

In the Brazilian groups, on the other hand, all but three of the correct solutions are **OKEQT**. In most cases the solution of the equation is

$$988 = 2x + 6$$
$$988 - 6 = 2x$$
$$982 = 2x$$
$$x = \frac{982}{2} = 491$$

or very similar. As we pointed out before, it is impossible to decide—in the absence of further explanation about the underlying model—whether this solution is guided by the "undo" perception linked to the "machine" model, by the perception of the whole-part relationship, or by a *numerical-arithmetical* model. In some cases, however, the solution of the equation involved steps that clearly characterise them as *numerical-arithmetical*, and the manipulation of the term involving the unknown characterises the *analiticity* of the solution, so those solutions are truly *algebraic* (Maurício N, AH8, who uses a *normal form* of the equation; Rogério C, AH8); in Maurício's explanation we have a further characterisation of the *analiticity* of his solution, as the unknown is treated explicitly as a number.

$$988 = 2x + 6$$
$$-2x - 6 + 988 = 0$$
$$-2x + 982 = 0$$
$$2x - 982 = 0$$
$$2x = 982$$
$$x = \frac{982}{2}$$
$$\boxed{x = 491}$$

SÃO 988 BRANCOS E EU MULTIPLIQUEI POR 2 O Nº DE PRETOS QUE É x. ~~AS~~ E SOMEI 6. O RESULTADO É O Nº DE PRETOS.

Maurício N, AH8

"There are 988 whites and I multiplied by 2 the no. of blacks and that is **x**. And added 6. The result is the number of blacks [sic]" (there should be no doubt from his script that he actually meant "the number of whites")

$$988 = 2 \times x + 6$$
$$988 = 2x + 6$$
$$-2x = +6 - 988$$
$$-2x = -982 \quad .(-1)$$
$$+2x = +982$$
$$x = \frac{+982}{2} \qquad x = 491$$

R = São necessários 491 azulejos pretos.

Rogério C, AH8

In another **OKEQT** solution (Andrea M, AH8), the evidence for an *algebraic* solution is direct from the explanation.

$$\boxed{b = 2p + 6}$$
$$988 = 2p + 6$$
$$2p + 6 = 988$$
$$2p = 988 - 6$$
$$2p = 982$$
$$p = 491$$

no enunciado tinha a fórmula. E também tinha o nº de azulejos brancos, então, foi só substituir na fórmula a variavel (nº de brancos) pelo número dado. E depois ~~fazer~~ separar variavel de número.

Andrea M, AH8

"in the statement there was the formula. And also the no. of white tiles, so, it was only a matter of substituting into the formula the variable (no. of whites) by the number given. And then to separate variable from number."

Three solutions — all coming from Brazilian students — treated the problem as one directly involving proportion, most probably suggested by the "8 whites for 1 black, etc." subtitles to the illustration[69]. Both Mariana O's (AH8) and Mairê M's (AH8) solutions are incorrect due to a mistaken perception of the relationship between the number of white and black tiles. Mariana's is clearly based on an algebraic model for solving the proportion: it is

[69]That no English student made this type of mistake suggests that the unfamiliarity of the Brazilian students with the problem also played a role.

numerical-arithmetical and *analytical,* with the focus being in determining the number of black tiles.

$$\frac{10}{2} = \frac{988}{x} \qquad 10x = 1976 \qquad x = 197{,}6$$

Eu descobri porque a 1ª fração tem que ser proporcional à segundo, a segunda à terceiro e assim vai...

Mariana O (AH8)

"I found out because the first fraction has to be proportional to the second, the third and so on..."

Mairê's solution, however, is *synthetical,* as the focus of the solution process is in determining the multiplier that multiplied by the number of black tiles in the simpler ratio (in this case, by 1) will produce the number of black tiles corresponding to 988 white tiles.

988 BRANCOS para ? PRETOS
8 BRANCOS PARA 1 PRETO
988 : 8 = 123.5 × 1
988 brancos para 123.5 pretos, mas não podemos cortar o azulejo então:
988 brancos para 124 pretos.

Se para 1 preto são 8 brancos (8 vezes a mais) é só saber quantos "8" tem 988 e multiplicar por 1 pois é 1 preto para 8 brancos

Mairê M (AH8)

Left: "988 whites for ? blacks(...)8 whites for 1 black (...)988 whites for 123.5 blacks, but we can't split the tile, so: 988 whites for 124 blacks."

Right: "If for 1 black there are 8 whites (8 times more), then it's only a matter of knowing how many '8' there are in 988 and multiply by 1, because it is 1 black for 8 whites"

Around a third of all **WCALC** mistaken solutions resulted from the incorrect use of the "reverse the formula" approach (Dawn H, FM3)[70].

~~988 ÷ 2 = 494 − 6 = 488~~
~~488 × 2 + 6~~ = 6
988 ÷ 2 = 494 − 6 = 488
488 + 6 = 494 × 2 = 988

Dawn H, FM3

In Laura G (AH7) we have a behaviour that is as close as one can get to a pure syntactical "shuffle": "white" and "black" are swapped, and the operations "reversed" without any regard for the *arithmetical articulation* or to the *meaning* of the resulting transformation within the *Semantical Field of numbers and arithmetic operations.* Nevertheless — and this is an important point in relation to *meaning* — from Laura's point of view not only the procedure enabled her to find out the answer in an acceptable way, but she was also able to correctly distinguish the symbols for the operations and associate them correctly with the symbols for the corresponding reverse operations; however, she has certainly not grasped the *intended meaning* that the teacher tried to convey.

nº de pretos = (2 ÷ nº brancos) − 6 (operação inversa)
(988)
(divisão)
nº pretos = 494 − 6
nº pretos = 488

Laura G, AH7

[70]It is interesting that at first she incorrectly applies the "reverse the formula" approach, not regarding the order in which the operations would be performed were the formula being used. When she tries to check the result against the original formula, it naturally does not work, but instead of rethinking the solution process, she alters the checking "template" to fit the mistaken solution procedure.

It is interesting that although the preferential approach to produce correct answers in AH7 was to solve the formula as an equation, more than three-quarters of the mistakes come from **WCALC** solutions, suggesting that even those solutions "by equation" might well have been guided by a contextualised model, as a failure to produce an algebraic model is strongly associated with a failure to produce a contextualised one.

The Salesp problem

As we expected, all the explanations provided with **OKCALC** solutions (which account for 77% of **all** answers) corresponded to the model "take away the fixed part from the total and see how many cars (or fridges) it corresponds to". The "explanation" for the initial subtraction is always a non-explanation (ie, "that's what you do"), and there was never any attempt to relate it explicitly to a whole-part relationship, the procedure being considered as self-justified (Fabíola, AH7); in a few scripts only there is a slight hint that the perception of a whole-part relationship might have guided those solutions (Aluízio A, AH7; Jacob B, FM3; Tarek S, AH7). Both Aluízio and Jacob seem to use a *comparison of wholes* strategy, while Tarek uses a *whole-part decomposition* model.

salario - 10.200,00
160,00 por cada geladeira

11 480
- 10.200
01 280 | 160
0 00 | 8

Vendeu 8 geladeiras

→ Ele ganha 10.200 de salário, então eu tirei 10 200 de 11.480 (quanto ele ganhou) o que sobrou evidentemente era das geladeiras e eu dividi por 160 (quanto ele ganha por cada uma)

Fabíola, AH7

She gets a 10,200 salary, so I took 10,200 from 11,480 (the money she earned) what is left is **evidently** [the money earned] because of the fridges..." (our emphasis)

11.480
- 10.200
1 280

1280 |160
000 8 geladeiras

R: Carla este mês vendeu 8 geladeiras

Explicação: se ela recebe 10.200 cruzeiros + 160 por cada geladeira (salário fixo) e este mês ela recebeu 11.480 eu tenho que tirar a diferença entre os dois salario para descobrir o quanto ela recebeu a mais e dividir pelo preço de cada geladeira para saber o nº de geladeiras

Aluízio A, AH7

"Explanation: if she got 10,200 + 160 for each fridge (fixed salary) and this month she got 11,480, then I have to calculate the difference between the two salaries to know how much she got **in excess** ..." (our emphasis)

I would take £185 off £360, giving £175, now all I have to do is find out how many £35 there are in £175, 175 ÷ 35 = 5, so Charles sold 5 cars that week.

I did it that way because I know he earns £185 fixed per week, anything over that is for cars he sells, so I took £185 off £360, and the answer (£175) must be divided by £35 to get the no. of cars, which is 5
he sold

Jacob B, FM3

11480 -
10200
01280
1280 | 160
00 8

Se o salário fixo de 10200 for
retirado do salário total deste mês,
restará apenas o ganho pelas gela-
deiras vendidas. Então é só dividir
o dinheiro das geladeiras pelo valor
de cada venda de geladeira que resta-
rá o nº de geladeiras

Tarek S, AH7

"If the fixed salary of 10,200 is taken from the total income there will be left only the [money] earned from the fridges..."

The focal point here is that in all three cases, the choice of subtraction is not informed by the *arithmetical articulation of an equation*, but by the need to evaluate parts produced through a decomposition of the whole, ie, the arithmetical operations are *tools* used to produce a required *evaluation*, and not informative *objects*. Nevertheless, a distinction between the approaches may be made, as the whole-part based model apparently guiding Aluízio's and Jacob's and Tarek's solutions is certainly more general.

Another illuminating aspect of the scripts, is that in 29% of the **OKCALC** solutions, the determination of the number of cars (fridges) sold is done using a number of different build-up and "build-down" strategies (Helen R, FM3; Derek G, FM2), and in those cases the *evaluation* of the "extra" money is not even considered, as the "fixed salary" (£185 in the English tests) is the target or the starting point, showing conclusively that those procedures are not "disguised" or "primitive" forms of division or multiplication.

1\. 360 − 35 = 325
2\. 325 − 35 = 290
3\. 290 − 35 = 255
4\. 255 − 35 = 220
5\. 220 − 35 = 185

Charles sold 5 cars

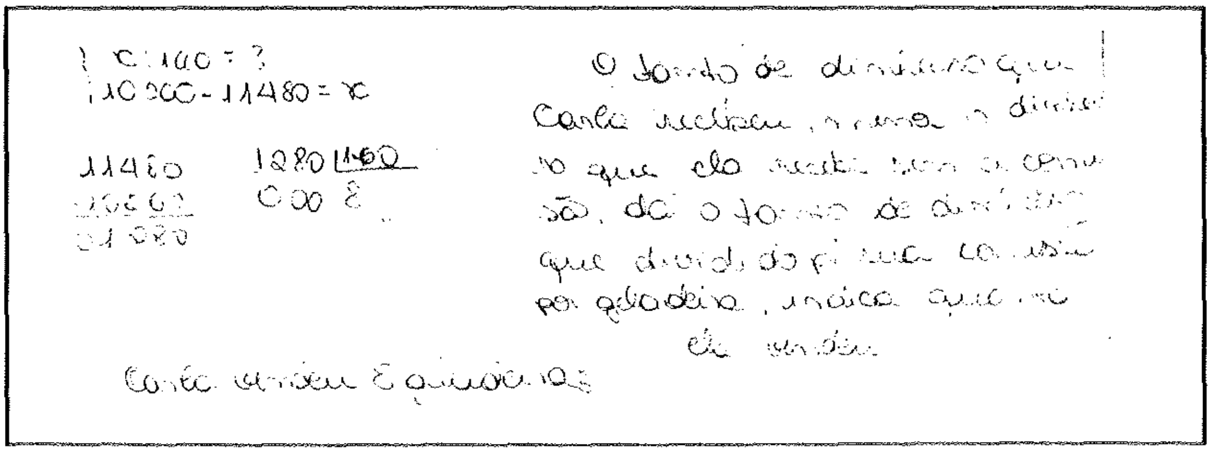
Helen R, FM3

He Sold 5 cars

I Put 185 on the calculator and added 35 5 times to the 185

Derek G, FM

Ana F (AH8) uses an "x", but her solution is clearly guided by the "selling" context, as the accompanying explanation shows; the "x" is used only to represent a value that can be immediately determined and is never manipulated before it is *evaluated.* It is suggested in the script that the focus of attention of the solution process seems to be the amount the saleswoman got for selling fridges, as Ana first writes "x+140=?," and this may be linked to the fact that as many students she saw the *evaluation* of the "extra" money as nothing more than evident and immediately possible.



Ana F, AH8

"The amount of money Carla got, minus the money she gets without the commission, gives the amount of money that divided by her commission by fridge indicates how many she sold."

The Secret problem

As we saw before, one relevant aspect in relation to this question was the unexpectedly high facility level, with the exception of FM2, which performed very badly.

The **OKEQT** solutions were in all cases solved by following the very standard

$$6x + 165 = 63$$
$$6x = 63 - 165 = -102$$
$$x = \frac{-102}{6}$$
$$x = -17$$

The one aspect of interest is that of all solutions employing equations, in only one case the solver correctly reached the third line then to produce an incorrect result (**+17**). When we compare this with the fact that many more similar mistakes were made in **Sec-** (analysed in the previous section), there is an indication that using a positive integer as a divisor makes more sense than using a negative one, possibly because the positive integer corresponds better to a "sharing" model of division, even if the amount being shared is negative; a further implication of this would be that the preference for *non-numerical* models (in this case the *analiticity* does not seem to be relevant) might be on the basis of some obstacles to the learning of the arithmetic of directed numbers.

In some of the **OKCALC** solutions (Elizabeth W, FM3, for example), the student considers that the answer **has to be** a negative number; however, as opposed to similar situations in solutions to **Sec-** (see previous group of problems), this consideration was never central to the process of solution, ie, it did not result in the transformation of the original problem into an auxiliary one.

To find the secret number I must do the same to each side. Really, the answer is going to be a minus because on it's own 63 is less than the extra 165. So I take 165 from 63 and get -102 so [6 x secret no.] = -102 so, -102 divided by 6 is -17.
I cannot be sure if this is correct!!!

Elizabeth W, FM3

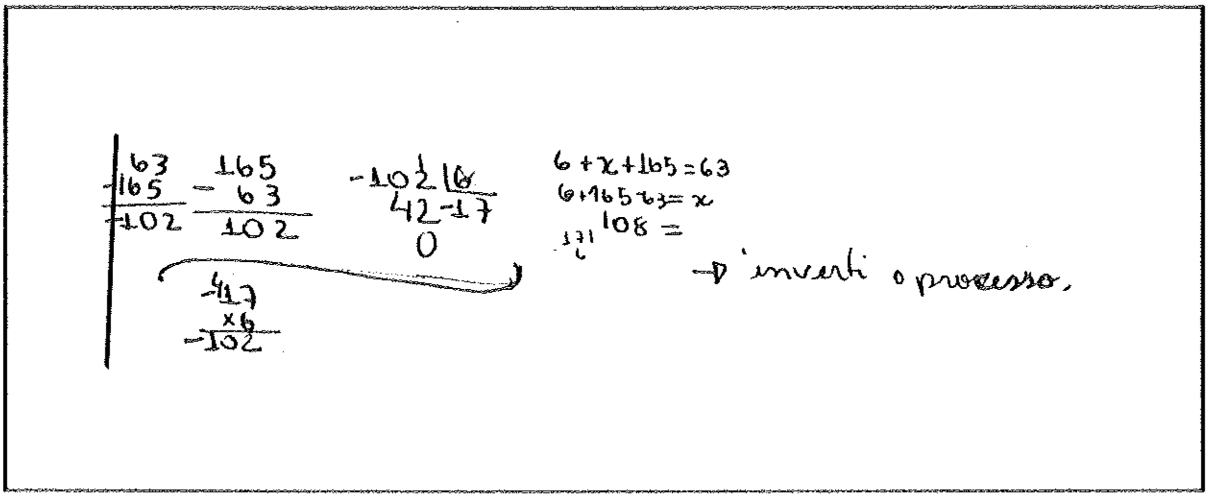
In one case the student concluded that the problem could not be done because adding would make always more than 165 (Jayne H, FM2).

I don't think that this can be done because 6 x n + 165 would be more than 63 e.g 6x2= 12+ 165 = 177 would it does not equal 63.

Jayne H, FM2

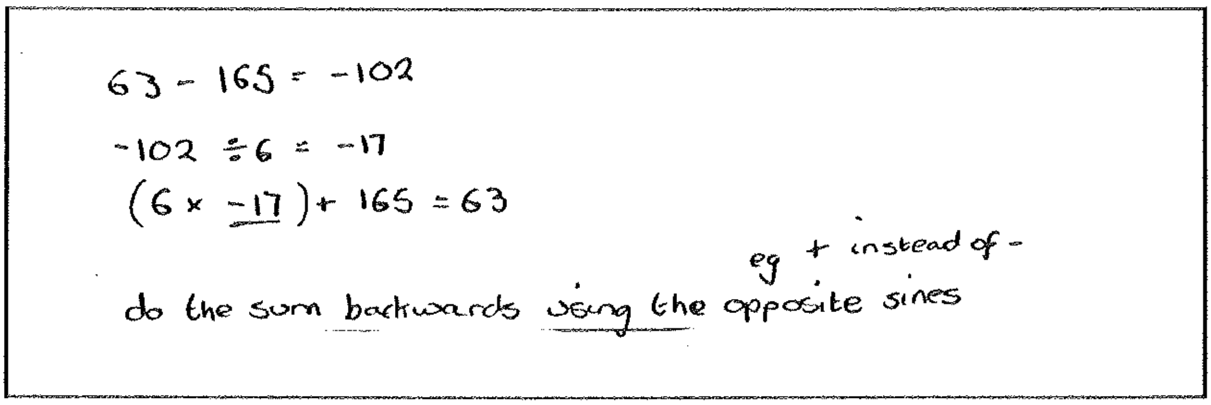
Jayne, however, failed to solve both **25-37** and **20-(-10)**, showing that her understanding — and possibly perception — of negative numbers was very weak. As a consequence, the distinction between using a *whole-part* model or a *numerical-arithmetical* one becomes somewhat blurred, as the *objects* in each of the two *Semantical Fields* have properties that are easily put into correspondence, or, put in a more precise way, it is easy to establish a much stronger *isomorphism* between the two *Semantical Fields* than in the general case. Nevertheless, and this is a central point in respect to the overall argument of our research work, it would be incorrect to characterise under those circumstances and on the basis of the possibility of the *isomorphism*, solutions using a *whole-part* model as involving *algebraic thinking*. The crucial point to produce the distinction is that arithmetical operations will still be used as tools only, while operations on the wholes and parts (joining, separating, etc) will be the *object operations*.

From the remaining **OKCALC** solutions, in all but two cases of an explanation being provided beyond a restatement of the calculations performed, they refer explicitly to "doing it backwards" or "reversing the process" (Camila A, AH7; Clare B, FM3; Hannah G, FM3; Shazia A, FM3).



Camila A, AH7

"I reversed the process"



Clare B, FM3

63
-165
-102

-102 ÷ 6 = -17

On this I worked backwords so first I found out what secret number was before adding 165. That was 63 - 165 = -102. Now I need something x 6 = -102. So I divided -102 by 6 = -17. Now -17 x 6 = -102 -102 + 165 = 63. Secret number = -17

Hannah G, FM3

~~165 - 63 = 102~~

63 - 165 ÷ 6 = -17   -17 is the secret number

Because you have the final answer you can find the missing answer by doing the sum the opposite way round

Shazia A, FM3

It is clear from those scripts that the resemblance with the "reverse the formula" procedure used by many students to solve **Patt** is strong. In Camila's script we have no further explanation, but Clare makes a distinction between "doing the sum backwards" — which seems to refer to the process of "going back" — and "using the opposite signs" — referring to the "undoing" of the effect of the operators, while Hannah specifically mentions that she "found out what secret number was **before adding 165**" (our emphasis), showing the "undo" intention. In Shazia's script the indication is even more complete, as she speaks of "the **final** number" (our emphasis), again a clear reference to a chain of calculations.

Given the reasonably high level of facility for this problem, and that, as we saw in respect to **Sec-** (see previous group of problems), the use of *whole-part* models with problems involving negative numbers is troublesome, we are led to think that most of the **OKCALC** solutions to this problem were guided by a *state-operator machine* model, as the one depicted in figure Patt 1. As we have already shown, this model develops within a *Numerical Semantical Field*, although it is not an *algebraic* model in this case for the lack of *analiticity*. The important implication of this result is that around 50% of all students answering this question were willing to operate within the *Semantical Field of numbers and arithmetical operations*. Moreover, it shows that this willingness is not the expression of a general, conscious, *conception*, but rather an implicit component of the procedure — either taught or developed — to deal with this specific type of problem.

Two other aspects are worth mentioning. First, that a *state-operator machine* model could be made to work with a problem like **Sec-** if *analiticity* becomes a part of the mode of thinking in which one is operating (see figure Patt 2)

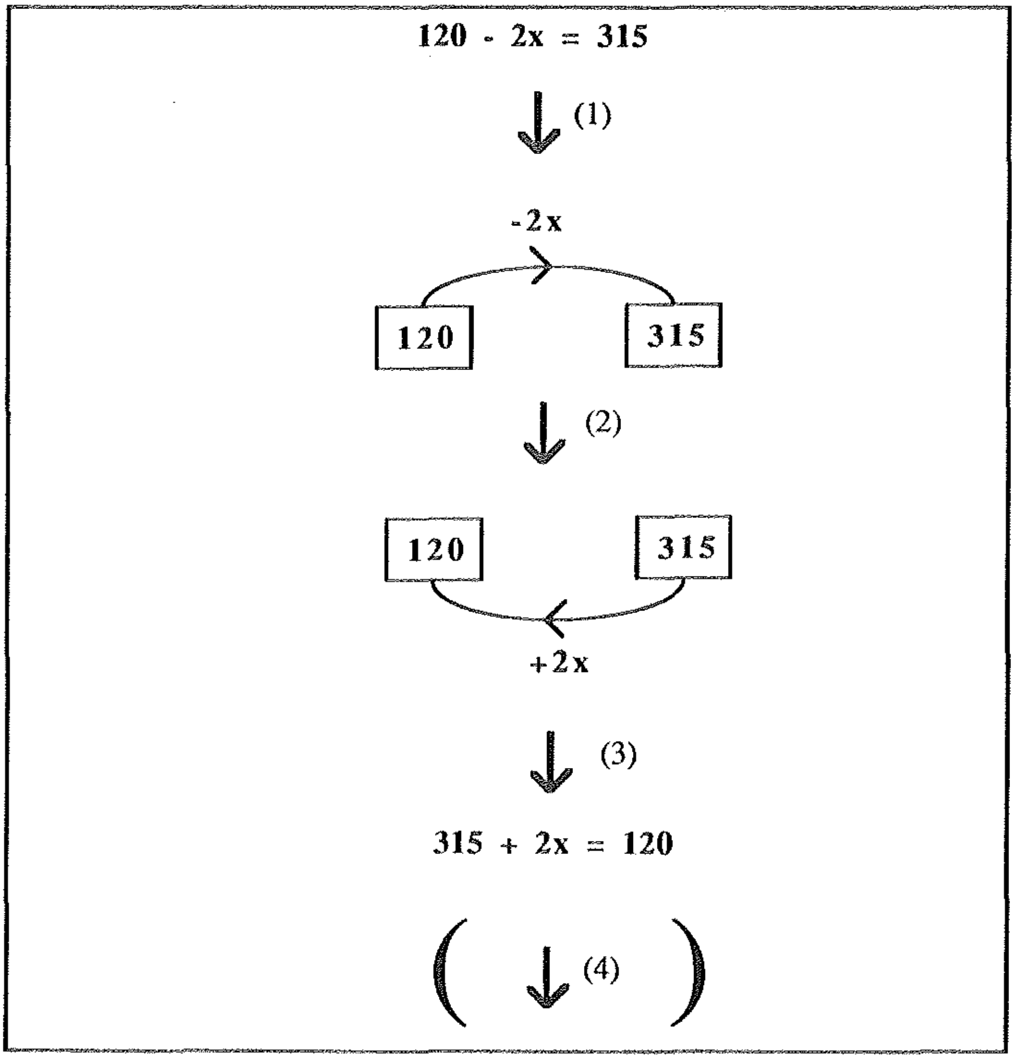


fig. Patt 2

Such approach has two merits: (i) it can be built entirely within the *Semantical Field of numbers and arithmetical operations*, from much simpler cases, and (ii) it introduces the notion of unknown with an *analytical* characteristic. A further advantage would be to strengthen the links between two useful forms of representation of *arithmetical articulation*, namely, the *state-operator diagram* and the standard algebraic notation. Step (4) in fig. Patt 2 could either be a return to a *state-operator model*, which would be similar to that used with **Secret**, or an algebraic solution of the equation, if the solver sees it as *meaningful*. In any case, steps (1), (2) and (3) alone might well serve as an alternative to a justification based on DSBS, for the transformation

$$\mathbf{120 - 2x = 315}$$
$$\mathbf{120 = 315 + 2x}$$

It must be clearly understood that we are not advocating this approach as a *panacea* that would provide the solution for all the problems involved in developing an *algebraic mode of thinking*, but it certainly is a strong and helpful paradigm from which other approaches may be developed.

SUMMARY OF FINDINGS AND CONCLUSION

The main point illustrated by the scripts to this group of problems is the possibility of a model that is clearly *numerical-arithmetical* but not *analytical*. Some solutions to problems in the previous groups had already presented this characteristic (for example, using a paradigmatic simpler example), but the use of a *state-operator machine model* highlighted the fact that it is possible for children in the age group we studied to accept a mode of thought that involves operating totally within the *Semantical Field of numbers and arithmetical operations*; this is particularly relevant because **Patt** is a problem where a spatial configuration is present, making clear that the problem is about numbers of tiles and not "pure" numbers, and yet many students used the *numerical-arithmetical model*. The use of a *state-operator machine model* also offers a singular illustration of the following points:

- arithmetical operators as *objects*, informing the manipulation process;
- the possibility of achieving some degree of *analiticity* in the process, by using generic or unknown parameters in the arithmetical operators (as in figure Patt 2)
- both *structure*—in the form of the *arithmetical articulation*—and *process*—in the form, for example, of the actual inversion of an operator, or of the actual chain of calculations—are indissoluble aspects of the manipulation of the model;

Structure in relation to the establishment and manipulation of a model is a notion that has to accommodate the possibility that there are *objects* that are not "formally" distinguished (eg, both the unknown and the parameters are seen as numbers) but neither there exists in the model a super-class containing both *objects* nor all properties applying to one such *object* applies to all of them (eg, in the "meaninglessness" of operating on or with the unknown). The *structure* of a model is, then, a *net of meanings*, necessarily local, and not an abstract and "clean" construction. Even when the establishment of a model is consciously informed by the knowledge of a more generic, general or abstract knowledge, it is only in the local sense of a *net of meanings* that the *structure* of the model is realised, and it is precisely in this sense that the term *arithmetical articulation* expresses the *structure of an algebraic expression as given by its composition in terms of numbers and arithmetical operations*..

Also, a solution to, say, **Patt**, using a *state-operator machine model* is structurally distinct from one using a *whole-part model* to model the "formula", and both are structurally distinct from the *analogical* solution that is based on a perception of the spatial configuration, and they are all structurally distinct from an *algebraic* solution employing an equation, although the *procedural* aspects may be similar.

4.7 Conclusions to the Chapter

The main result of the experimental study was to confirm that there are different models underlying students' solutions; moreover, it has also shown that our distinction between *algebraic* and *non-algebraic* solutions, based on our characterisation of *algebraic thinking*, offers a clear and useful framework for distinguishing and characterising those solutions.

From the point of view of the methodology adopted—using groups of related problems, instead of "isolated" items—proved to be a correct and very useful choice, as many of the aspects of the models that were identified could only be clearly understood by comparing its use in problems with different contexts and with different numerical parameters. The decision of not using interviews meant we could not probe in depth some aspects of the underlying models, but, on the other hand, it reassured us that it is indeed possible to understand much of those underlying models by examining only pupils' written work, an important feature of the methodology, both because of the possibility of carrying out studies with a larger number of pupils, but also for the teacher who, many times, does not have the necessary time to accompany closely the discussion that goes on on each group during classroom activity.

The most problematic aspect for the students in our study, was that for those unable to deal algebraically with the *secret number* problems, the process of modelling them into a non-algebraic model proved to be an impossible, or at least, very difficult, task. The fact that most of those students could cope with the "contextualised version" of those *secret number* problems, led us to conclude that two are the probable sources of difficulties in the case of those *secret number* problems: (i) difficulties in interpreting the elements of the *arithmetical expressions* in terms of other models; particularly in the case of *whole-part* models, expressions of the type

$$\mathbf{ax + b = c} \quad \text{and} \quad \mathbf{b + ax = c}$$

were easier to interpret than expressions of the type

$$\mathbf{b - ax = c}$$

We suggest that this was the case because the former provide a much more direct representation of "a whole and its parts," while in the case of the latter, the elements have to be separately identified, and the *whole-part articulation* constructed; and (ii) this difficulty is only enhanced by the fact that the notion of a general *whole-part* model seems to be to a great extent alien to what those students see as knowledge applicable to those problems; as a consequence, making sense of the "decontextualised" *secret number* problems implied, in each case, looking for an adequate interpretation, possibly in terms of another problem with a "story," possibly in terms of experience with "plain calculations."

Another relevant aspect we were able to identify, was the importance of what we called *pointers*, in the manipulation of non-algebraic models, for example the fact that one should not add a weight with a length, or that a seesaw will be balanced only if equal weights are put on each side. As we have already pointed out, but wish to stress, this aspect suggests that the use of non-algebraic models to facilitate the learning of specific aspects of algebra—for example the scale balance—has to be carefully examined, in order to avoid the association of the algebraic procedures learned with those *pointers*, an association which may, and probably will, constitute a huge obstacle for the development of an algebraic mode of thinking, particularly in the case of "concrete" models.

From a more general point of view, it became clear that the central notion being examined in our study was that of *meaning*. In this sense, the distinction we used between *elements* of the problem and *objects* of the model, proved very helpful in highlighting the choice and interpretation of the *elements* of the problem which is involved in the process of establishing and manipulating a model.

The non-algebraic models we have identified in the scripts almost always involved an underlying *whole-part articulation*. Hypothetical manipulation of the context of the problem and geometric models appeared only in very few scripts.

The *state-operator machine* model, which appeared only in the Pattern group of problems, represents a special case, as it is clearly a *numerical* but *non-algebraic* model, as it lacks *analiticity*. The fact it was used by so many students, suggest that operating within a purely *numerical* environment, and using the arithmetical operations as *objects*, ie, manipulating a model *informed* by them, is not beyond the grasp of those students, supporting our claim that the development of an algebraic mode of thinking has to be understood as the process of cultural immersion from which the development of an *intention* is produced, and a process that is very much dependent on the exposure to that mode of thinking. The fact that among Brazilian students we were able to find many more instances of algebraic models being used than among English students, also supports this claim, given the distinct emphasis on the teaching of algebra—much greater in Brazil—in the grades in question.

Chapter 5
General Discussion

Both the evidence from the historical study and from the experimental study showed that our characterisation of algebraic thinking—*arithmeticity*, *internalism*, and *analiticity*—provides an adequate framework for distinguishing different ways of modelling problems and of manipulating those models. Moreover, we have also shown that by distinguishing those different modes of thinking, we were able to identify the tensions underlying the production of an algebraic knowledge, as well as the sources of the difficulties faced by the students in our experimental investigation and the constraints acting upon the development of an algebraic knowledge in historically situated mathematical cultures.

The central issue which provided the thread followed in our investigation is that of *meaning*. We identified two ways in which the issue of *meaning* is related to our study of *algebraic thinking*.

First, an "algebraic verbal problem" can be seen either as the problem of determining the required measure(s) or as the problem of determining a number or numbers which satisfy some given arithmetical conditions; in the case of "purely numerical problems," interpreting it as the problem of determining a measure requires the extra step of interpreting the elements in the "arithmetical" statements—as, for example, in the *secret number* problems in our test papers—as representing or describing some contextualised problem[1]. The fact that *secret number* problems were consistently more difficult than the corresponding contextualised problems—apart from the case of the older Brazilian students, who had had a somewhat thorough experience with using equations to solve problems—indicates that for the students in our experimental study, interpreting the "arithmetical" statements into another *Semantical Field* was not an easy task; both the lack of the pointers we have mentioned in Chapter 4—eg, "weights can only be added to or subtracted from, other weights"—and the lack of taught *whole-part* models, which could provide a more or less standard *Semantical Field* for interpreting the "arithmetical" statements, seem to account for the failure of so many students to make sense of those statements.

The second way in which *meaning* is related to *algebraic thinking*, is through the process of manipulating the model used with a problem. Even if a problem is seen as the problem of determining a number or numbers which satisfy given conditions, the conceptions involved in the determination of the concept of *number* play a central role

[1]The quotes in *arithmetical* are necessary for this precise reason: as the solver makes sense of the statements by interpreting them in a *Semantical Field* other than that of numbers and arithmetical operations, we may safely assume that those statements are not seen *primarily* as arithmetical statements; this does not imply, however, that the solver is *intelectually incompetent* to do so, but only that within his or her mathematical culture that is not the preferential mode of thinking.

in determining what can and should be done to manipulate relationships involving *number*; the historical study provided precisely the evidence about how conceptualisations of number are central if we are to understand the mathematical activity within a mathematical culture—or of an individual. We have clearly shown that *algebraic thinking* depends on a *symbolic* understanding of numbers, but also that such a *symbolic* understanding of numbers have to compete with other—quite acceptable—conceptions, such as "number as measure." The tension between a *symbolic* understanding of number, which implies that numerical-arithmetical relations are treated *arithmetically, internally,* and *analytically,* ie, algebraically, and an *ontology* of *number*, which says what number *is* and only from there one determines how it can be dealt with, is a central issue in the process of developing an algebraic mode of thinking; our experimental study did not intend to probe into the students' mathematical conceptions underlying their mathematical activity, but nonetheless, it provided evidence that the models underlying their solutions to the proposed problems did not present—in many cases—the generality *as a method* that Jacob Klein indicates as the central aspect distinguishing Vieta's conceptualisation of algebra from that of Diophantus, and which is a central characteristic of what he calls the "modern" conceptualisation of the mathematical activity.

Those two aspects of the relationship between *meaning* and *algebraic thinking* suggest a focus of tension in the development of an algebraic mode of thinking. The acceptance of the "arithmetical" statements as informative in themselves, ie, as true arithmetical statements, certainly depends on the possibility of treating them *algebraically*, at the same time thinking algebraically depends on the ability to recognise arithmetical statements as informative in their own right. Our approach to this question was to consider *algebraic thinking* as an *intention*, more precisely, the *intention* to treat problems which involve the determination of a number or numbers *algebraically*, according to our characterisation of *algebraic thinking*; the *intention* to think algebraically can certainly evolve from very simple algebraic situations, such as solving simple equations, but precisely because this *intention* is not algebra, only a way of dealing with algebra, *the production of an algebraic knowledge, eg, "how to solve equations of a certain type," does not depend on or involves* ***by itself*** *algebraic thinking*. It is only by making that *intention* explicit, and by contrasting *algebraic thinking* with other modes of thinking which can be used to produce algebra, that the *intention* of thinking algebraically can be consciously acquired. Moreover, it is only when such *intention* is in place that the requirement of a treating arithmetical statements in a way which is *arithmetical, internal,* and *analytical,* can be meaningful.

In the course of our investigation of the nature of *algebraic thinking*, two important distinctions were elicited: (i) that between *intrasystemic* and *extrasystemic meaning*; and, (ii) that between *situational* and *mathematical context.*

The former allows us to account for the possibility of an *algebraic* algebraic activity (as opposed to a *non-algebraic* one), by making clear that, far from being *meaningless*, or *semantically weak*, the elements involved in *algebraic thinking* are *meaningful* and *semantically full*, but only when interpreted within the *Semantical Field* of numbers and arithmetical operations, ie, there is a shift of referential which makes the *algebraic* algebraic activity meaningful. In the historical study we had the opportunity to refer to the *syntactical meaning* of the elements in *algebraic thinking.* This notion, which might seem paradoxical at first, is essential for one to understand what *algebraic thinking* is, and must be accepted not as a linguistic detour to indicate the usually accepted notion of "rule manipulation," be it in a poorly or in a highly skilful manner, but as indicating that there is nothing "outside" the statements being manipulated which are required to make their elements "meaningful."

The second of the two distinctions allows us to understand the importance of one's willingness to shift into a new *Semantical Field* in the process of thinking algebraically. It is the shift from the *situational context* of a problem—or from its *local context* in the case of "purely numerical problems"—into a *mathematical context,* representing also the transition from the problem to a method for solving the problems of a class to which the specific problem in question belongs, or seems to belong, that makes *algebraic thinking* possible; moreover, the very *intention* of producing that shift—and, thus, its acceptance—is that which characterises mathematics as an accepted cultural object. The refusal by Luria's and by Freudenthal's subjects to operate within a "context-free" environment strongly indicates that the development of a given mathematical mode of thinking depends on the acceptance of the fact that certain ways of organising the world are adequate and useful, ie, that they produce insights which conform to one's cultural needs. It is exactly in this sense that *algebraic thinking* has to be understood as an *intention*: it represents the affirmation of the need to use numerical-arithmetical models and to treat those models *arithmetically, internally*, and *analitically*, and it is by affirming this need that it drives the development of an algebraic knowledge.

By understanding *algebraic thinking* as a cultural component, rather than a developmental one, we opened a line of research into the difficulties faced by children in the learning of algebra; we have shown that non-algebraic models used as primary ways of dealing with problems involving the determination of a number or numbers do constitute an obstacle to the development of an algebraic mode of thinking, and we have elicited some of those models and their main characteristics. By also showing that *algebraic thinking* is better understood as an *intention,* we demonstrated that the process of developing an algebraic mode of thinking is one of *cultural immersion,* and by doing so, we open the possibility of explaining the "failure" of individuals in "naturally" developing the ability to think algebraically—as Piaget's theory, for example, would predict—in terms of a *lack of a cultural component.* In a similar way, we think that it is possible to explain, for example, the "failure" of individuals in "naturally" developing *proportional reasoning.*

At a deeper level, this aspect of our investigation shows, in particular in relation to the historical study, that asserting a parallel between the historical development of algebra and *algebraic thinking* and the development, by individuals, of an algebraic mode of thinking, cannot be understood in the context of searching for similar "stages of development." The cultural factors are, we believe, too complex to be "read through," and it thus seems to be the case that even if an underlying, inevitable, cognitive engine exists—as Garcia and Piaget say—we are unlike ever to reach it. The culturalistic approach, on the other hand, highlights knowledge as the result of trying to make sense of the world, and as the world is presented to us largely through the culture we live in, and as cultures are in perpetual recreation, the culturalistic approach to the nature of *algebraic thinking* provides an immediate understanding of the cultural process of being initiated to it.

Although our research has been thoroughly concerned with characterising *algebraic thinking,* one of its clearest results was to reveal the interplay between algebraic and non-algebraic modes of thinking. First, because non-algebraic models can provide, as in Davydov's teaching programme, the raw material which is to be examined algebraically; second, and more important, because the deep distinction between algebraic and non-algebraic modes of thinking point out to the impossibility of reducing one to the other, ie, it points out to the inadequacy of substituting algebraic for non-algebraic "whenever possible"; *algebraic thinking* can only be understood in the context of all different modes of thinking, and, thus, the development of non-algebraic modes of thinking has to be kept as a central objective of teaching. The possibility of interpreting a problem or situation within different *Semantical Fields,* certainly offers a richer perspective for organising one's world and for producing knowledge.

The results of our investigation point out, although in a provisional manner, that an early introduction of children to *algebraic thinking* should be carried out. First, because it provides a unifying and powerful *mathematical context,* one in which a deeper understanding of the structure of large classes of problems is possible. Second, because it allows the development of an understanding of numbers and of the arithmetical operations which is algebraic—and, thus, symbolic—from very early stages of learning, resulting in a much sounder mathematical foundation to those aspects of the children's mathematical knowledge. Third, because *situational* models and abstract non-algebraic models (eg, whole-part models) are a much more present part of everyone's life, and opportunities for refining and discussing them are much more abundant; emphasising the importance of algebraic models, particularly to the teacher and curriculum developer, is a proper way of restoring a balance which is necessary. Fourth, and finally, the traditionally accepted view of "algebra as generalised arithmetic"—under the guise of "numbers first and then algebra"—leads in fact to the formation of sometimes insuperable obstacles to learning, and an early start with *algebraic thinking* would address this difficulty.

There are two natural directions to follow after the research presented in this dissertation, both of which we will pursue.

The first is to extend our research into the history of mathematics, by examining other historically situated cultures and by considering the non-mathematical characteristics of the cultures examined. This last aspect is particularly important to provide a more comprehensive view of the place of the mathematical cultures in their "parent" cultures.

Second, we will study, this time making extensive use of interviews, students' conceptualisations in mathematics, particularly in relation to elements related to *algebraic thinking*. At the same time, we will engage in developing a teaching approach for the development of *algebraic thinking* in the later years of primary school and early years of secondary school; some of the exploratory work in this respect has already been conducted, both in Brazil and in England, and will be reported elsewhere.

Annex A
Problems used in the exploratory experimental study

1) Two friends, Maggie and Sandra, went to the Goose Fair.
Maggie brought £12 with her and Sandra brought £18.
During the afternoon, Sandra spent twice as much as Maggie, and when they left the fair, both of them had the same amount of money.
How much didi each of them spend?

2) A car salesman earns, per week, a fixed £200 <u>plus</u> £35 for each car sold.
This week his total income was £375.
How many cars did he sell this week?

3) A carpenter wants to cut a 73 cm long stick in two, but he wants one of the pieces to be 17 cm longer than the other.
How long will the pieces be?

4) I have a 'secret' number in my mind.
If I multiply it by three, and take the result away from 210, I'm left with 156.
Now, which is my 'secret' number?

5) Pick up any three consecutive numbers and write them down inside the squares.
Now add them up and put the result inside the circle.
Finally, divide the number in the circle by three and put this last result in the triangle.

An example:

$\boxed{12} + \boxed{13} + \boxed{14} = 39$ $\qquad$ $39 \div 3 = 13$

Now try with other successive numbers.

(a) will the number in the triangle always be equal to the middle number in the squares?

(b) Please explain how do you know that your answer to (a) is correct.

6) Johanne bought some bottles of milk and paid for it with a £5 note.

(a) can you work oput the change she received?

(b) If not, what else should you know to be able to work out the change?

7) Suppose you buy two chocolate bars, you pay for it and you get the change.

Then you decide to buy a can of cola.

When you are o pay, the clerk says: "Give me back your cahnge and I'll give you back your money. Now I add up the prices for the chocolates and the cola and you pay for the whole sum."

Is this the same as just paying, from the cahnge, for the cola?

Please explain your answer.

Annex B
Problems used in the main experimental study

Question 1

I am thinking of a secret number.
I will only tell you that

120 - (13 x secret no.) =315

The question is: Which is my secret number?
(Explain how you solved the problem and why you did it that way)

Question 2

To know the number of oranges that will be in a box, one has to divide the total number of oranges by the number of boxes, that is,

(oranges per box) = (number of oranges) ÷ (number of boxes)

a) There are 1715 oranges and we want to have 49 oranges per box. How many boxes are needed?

b) If you are told the number of oranges per box and the number of boxes, how would you work out the total number of oranges?

Question 3

From a tank filled with 745 litres of water, 17 buckets of water were taken. Now there are only 626 litres of water in the tank.

How many litres does a bucket hold?
(Explain how you solved the problem and why you did it that way)

Question 4

Maggie and Sandra went to a records sale.
Maggie took 67 pounds with her, and Sandra took 85 pounds with her (a lot of money!!).

Sandra spent four times as much money as Maggie spent.
As a result, when they left the shop both of them had the same amount of money.

How much did each of them spend in the sale?
(Explain how you solved the problem and why you did it that way)

Question 5

Mr Sweetmann and his family have to drive 261 miles to get from London to Leeds.
At a certain point they decided to stop for lunch.
After lunch they still had to drive 2.7 times as much as they had already driven.

How much did they drive after lunch? And before?
(Explain how you solved the problem and why you did it that way)

Question 6

a) 25 – 37 = ...

b) 20 - (-10) = ...

Test paper A1

Question 1

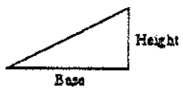
I am thinking of a "secret" number.
I will only tell you that ...

181 - (12 x secret no.) = 97

The question is: Which is my secret number?
(Explain how you solved the problem and why you did it that way)

Question 2

The slope of a ramp is calculated by dividing the height of the ramp by the lenght of its base. That is,

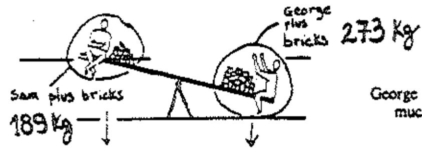


slope = height ÷ base

a) If the slope of a ramp is 1.2 and its base measures 15 metres, what is the height of this ramp?

b) If you are given the slope and the height of a ramp, how would you work out the base of this ramp?

Question 3



George throws away four times as much weight as Sam does.

Now they are balanced.

How many kilograms did George throw away? And Sam?
(Explain how you solved the problem and why you did it that way)

Question 4

On a TV show...
"Well, Mrs Swetmann! You have so far won 731 pounds in our show...
Now I have an offer for you:

CHOICE A: We multiply your prize by 1.2 and then we multiply the result by ... (and the presenter whispered a number in Mrs Sweetman's ear) ... or...
CHOICE B: the other way around: we first multiply your prize by the number I have just whispered to you, and then we multiply the result by 1.2 ... "

What would your choice be? (Justify your answer)

Question 5

John is organizing a big party for children.
He bought a 8 big boxes of candies, each one containing 250 candies.

If 250 children show up to the party, how many candies will each of them
(Evereybody gets the same number of candies, of course!)

Explain very clearly how you solved this problem.

Question 6

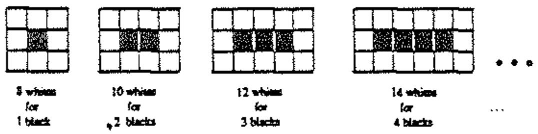
Sam and George bought tickets to a concert.
Because Sam wanted a better seat, his ticket cost four times as muc
George's ticket.
Altogether they spent 74 pounds on the tickets.

What was the cost of each ticket?
(Explain how you solved the problem and why you dit it that way)

Test paper A2

Question 1

Her you have a pattern of tiles:



One possible formula that gives the number of white tiles that go with a cert number of black tiles is:

no. of whites = (2 x no. of blacks) + 6

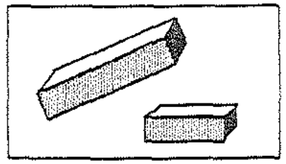
How many black tiles are needed, if I want to use 988 white tiles?
(Explain how you solved the problem and why you did it that way)

Question 2

At the right you have a sketch of wooden blocks.

A long block put together with two of the short blocks measure 162 cm altogether.

If two short blocks are put together, they still measure 28 cm less than a long block.



What is the lenght of each individual block?
(Explain how you solved the problem and why you did it that way)

Question 3

I am thinking of a "secret" number.
I will only tell you that ...

181 - (12 x secret no.) = 128 - (7 x secret no.)

The question is: Which is my secret number?
(Explain how you solved the problem and why you did it that way)

Question 4

Sam and George bought tickets to a concert.
Because Sam wanted a better seat, his ticket cost 2.7 times as much as George's ticket.
Altogether they spent 74 pounds on the tickets.

What was the cost of each ticket?
(Explain how you solved the problem and why you dit it that way)

Question 5

I am thinking of two secret numbers.
I will only tell you that...

(first no.) + (second no.) = 185
and
(first no.) - (second no.) = 47

Now, which are the secret numbers?
(Explain how you solved the problem out and why you did it that way)

Question 6

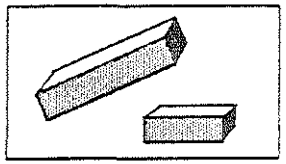
a) 25 – 37 = ...

b) 20 - (-10) = ...

Test paper B1

Question 1

At the right you have a sketch of wooden blocks.
A long block and a short block measure 162 cm altogether.
A short blocks measures 28 cm less than a long block.



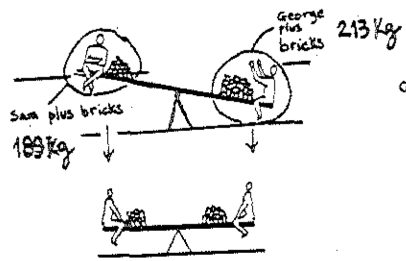
What is the lenght of each individual block?
(Explain how you solved the problem and why you did it that way)

Question 2

Mr Sweetmann and his family have to drive 261 miles to get from London to Leeds.
At a certain point they decided to stop for lunch.
After lunch they still had to drive four times as much as they had already driven.

How much did they drive before lunch? And after lunch?
(Explain how you solved the problem and how you knew what to do)

Question 3



George throws away 11 bricks and Sam throws away 5 bricks.

Now they are balanced.

What is the weight of one brick ?
(Explain how you solved the problem and why you did it that way)

Question 4

I am thinking of a "secret number".
I will only tell you that

$$(6 \times \text{secret no.}) + 165 = 63$$

The question is: Which is my secret number?
(Explain how you solved the problem and why you did it that way)

Question 5

Charles sells cars, and he is paid weekly.
He earns a fixed £185 per week, plus £35 for each car he sells.

This week he was paid a total of £360.

How many cars did Charles sell this week?
(Explain how you solved the problem and why you did it that way)

Test paper B2

Question 1

I am thinking of a "secret" number.
I will only tell you that ...

$$181 - (12 \times \text{secret no.}) = 128 - (7 \times \text{secret no.})$$

The question is: Which is my secret number?
(Explain how you solved the problem and why you dit it that way)

Question 2

Sam and George bought tickets to a concert.
Because Sam wanted a better seat, his ticket cost four times as much as George's ticket.
Altogether they spent 74 pounds on the tickets.

What was the cost of each ticket?
(Explain how you solved the problem and why you dit it that way)

Question 3

To know the number of oranges that will be in a box, one has to divide the total number of oranges by the number of boxes, that is,

(oranges per box) = (number of oranges) ÷ (number of boxes)

a) If there are 17 oranges per box and we have 49 boxes, how many oranges there are altogether?

b) If you are told the number of oranges per box and the total number of oranges, how would you work out the number of boxes needed?

Question 4

At Celia's shop you can buy boxes of chocolate bars or you can buy spare bars as well.

A box and three spare bars cost £8.85.
A box with three bars missing cost £5.31

What is the price of a box of chocolate bars in Celia's shop? What is the price of a single bar?
(Explain how you solved the problem and why you dit it that way)

Question 5

Abigail is having a hard time to decide what to dress.
She has socks of 6 different colours, skirts of 5 different colours, and T-shirts of 7 different colours.

In how many different ways can she dress? (Explain how you solved the problem and why you did it that way)

Test paper C1

Question 1

Maggie and Sandra went to a records sale.
Maggie took 67 pounds with her, and Sandra took 85 ponds with her (a lot of money!!).

Sandra bought 11 Lp's, and Maggie bought 5 Lp's.
As a result, when they left the shop both of them had the same amount of money.
What is the price of an Lp?

(Explain how you solved the problem and why you did it that way)

Question 2

Mr Sweetmann and his family have to drive 261 miles to get from London to Leeds.
At a certain point they decided to stop for lunch.
After lunch they still had to drive 2.7 times as much as they had already driven.

How much did they drive before lunch? And after lunch?
(Explain how you solved the problem and why you did it that way)

Question 3

I am thinking of two secret numbers.
I will only tell you that...

(first no.) + (3 x second no.) = 185
and
(first no.) - (3 x second no.) = 47

Now, which are the secret numbers?
(Explain how you solved the problem and why you did it that way)

Question 4

The speed of a car can be calculated by dividing the distance covered by the time spent to do it. That is,

speed = distance ÷ time

a) If one has to travel 351 kilometres at a speed of 110 kilometres per hour, how much time will it take?

b) If you are told the speed of a car and the amount of time it ran, how would you work out the distance it covered?

Question 5

Joe's Cafe offers a number of choices of bread, fillings and sauces. There a[re] 84 different combinations altogether.

A customer counted 14 different sauces on the menu.

If one wants only bread and filling, how many choices are available?
(Explain how you solved the problem and why you did it that way)

Test paper C2

TICKET AND DRIVING

	AH7				AH8				FM2				FM3			
	T4	D2.7	D4	T2.7	T4	D2.7	D4	T2.7	T4	D2.7	D4	T2.7	T4	D2.7	D4	T2.7
	40	40	16	16	34	34	19	19	36	36	17	17	41	41	25	25
OKEQT	0.42	0.15	0.13	0.13	0.73	0.29	0.42	0.53	0.03	0.05	0.00	0.00	0.02	0.00	0.00	0.00
OK ÷3.7 or 5	0.43	0.00	0.32	0.06	0.15	0.03	0.31	0.00	0.33	0.06	0.41	0.00	0.68	0.10	0.80	0.16
OKT&E	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.06	0.02	0.00	0.04	0.20
W ÷2.7 or 4	0.02	0.23	0.13	0.25	0.06	0.09	0.00	0.11	0.20	0.14	0.06	0.00	0.07	0.32	0.08	0.20
WOTH	0.05	0.20	0.19	0.38	0.05	0.39	0.11	0.26	0.20	0.14	0.29	0.29	0.12	0.27	0.04	0.28
NATT	0.08	0.43	0.25	0.19	0.03	0.21	0.16	0.11	0.25	0.61	0.24	0.65	0.07	0.31	0.04	0.16
OK	0.85	0.15	0.44	0.19	0.88	0.32	0.74	0.53	0.36	0.11	0.41	0.06	0.72	0.10	0.84	0.36
WRONG	0.07	0.43	0.31	0.63	0.11	0.48	0.11	0.37	0.40	0.28	0.35	0.29	0.19	0.59	0.12	0.48
NATT	0.08	0.43	0.25	0.19	0.03	0.21	0.16	0.11	0.25	0.61	0.24	0.65	0.07	0.31	0.04	0.16

Annex C
Data on the groups in the main experimental study

Group: AH7 (Brazilian 7th graders)

Total no. of students: 56

Average age (yrs.mths): 13.11

Standard deviation (yrs.mths): 0.9

Group: AH8 (Brazilian 8th graders)

Total no. of students: 53

Average age (yrs.mths): 15.0

Standard deviation (yrs.mths): 1.0

Group: FM2 (English 2nd year)

Total no. of students: 53

Average age (yrs.mths): 13.2

Standard deviation (yrs.mths): 0.4

Group: FM3 (English 3rd year)

Total no. of students: 66

Average age (yrs.mths): 14.3

Standard deviation (yrs.mths): 0.3

Group: ALL

Total no. of students: 228

Average age (yrs.mths): 14.1

Standard deviation (yrs.mths): 0.11

Observation: In Brazilian groups, the much greater standard deviation is due to the fact that students can actually fail a whole year, which does not happen in English schools.

Annex D

Tables of frequencies for the problems in the main experimental study

TICKET AND DRIVING

	AH7				AH8				FM2				FM3			
	T4	D2.7	D4	T2.7	T4	D2.7	D4	T2.7	T4	D2.7	D4	T2.7	T4	D2.7	D4	T2.7
	40	40	16	16	34	34	19	19	36	36	17	17	41	41	25	25
OKEQT	0.42	0.15	0.13	0.13	0.73	0.29	0.42	0.53	0.03	0.05	0.00	0.00	0.02	0.00	0.00	0.00
OK ÷3.7 or 5	0.43	0.00	0.32	0.06	0.15	0.03	0.31	0.00	0.33	0.06	0.41	0.00	0.68	0.10	0.80	0.16
OKT&E	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.06	0.02	0.00	0.04	0.20
W ÷2.7 or 4	0.02	0.23	0.13	0.25	0.06	0.09	0.00	0.11	0.20	0.14	0.06	0.00	0.07	0.32	0.08	0.20
WOTH	0.05	0.20	0.19	0.38	0.05	0.39	0.11	0.26	0.20	0.14	0.29	0.29	0.12	0.27	0.04	0.28
NATT	0.08	0.43	0.25	0.19	0.03	0.21	0.16	0.11	0.25	0.61	0.24	0.65	0.07	0.31	0.04	0.16
OK	0.85	0.15	0.44	0.19	0.88	0.32	0.74	0.53	0.36	0.11	0.41	0.06	0.72	0.10	0.84	0.36
WRONG	0.07	0.43	0.31	0.63	0.11	0.48	0.11	0.37	0.40	0.28	0.35	0.29	0.19	0.59	0.12	0.48
NATT	0.08	0.43	0.25	0.19	0.03	0.21	0.16	0.11	0.25	0.61	0.24	0.65	0.07	0.31	0.04	0.16

SEESAW-SALE-SECRET NUMBER ("E"="Seesaw"; "A"="Sale.")

	A7					AH8					FM2					FM3				
	E11-5	E4x	A11-5	A4x	SecNo	E11-5	E4x	A11-5	A4x	SecNo	E11-5	E4x	A11-5	A4x	SecNo	E11-5	E4x	A11-5	A4x	SecNo
	16	21	19	21	35	19	17	17	17	36	17	20	16	20	33	25	24	17	24	42
OKEQT	0.06	0.14	0.05	0.24	0.40	0.16	0.47	0.35	0.47	0.88	0.00	0.05	0.00	0.10	0.00	0.00	0.04	0.00	0.00	0.10
OKCALC	0.13	0.00	0.16	0.00	0.03	0.00	0.00	0.06	0.00	0.00	0.06	0.00	0.00	0.00	0.03	0.44	0.08	0.35	0.04	0.05
OKT&E	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.12	0.00	0.38	0.15	0.00	0.20	0.13	0.24	0.33	0.00
WEQT	0.13	0.10	0.05	0.05	0.09	0.32	0.12	0.18	0.24	0.06	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.04	0.10
WCALC	0.25	0.43	0.31	0.43	0.08	0.10	0.12	0.12	0.06	0.03	0.41	0.60	0.38	0.20	0.37	0.24	0.42	0.24	0.37	0.43
NATT	0.44	0.33	0.42	0.29	0.40	0.42	0.29	0.29	0.24	0.03	0.41	0.35	0.25	0.55	0.58	0.12	0.33	0.18	0.21	0.31
OK	0.19	0.14	0.21	0.24	0.43	0.16	0.47	0.41	0.47	0.88	0.18	0.05	0.38	0.25	0.04	0.64	0.25	0.59	0.38	0.15
WRONG	0.38	0.52	0.37	0.48	0.17	0.42	0.24	0.29	0.29	0.09	0.41	0.60	0.38	0.20	0.37	0.24	0.42	0.24	0.42	0.53
NATT	0.44	0.33	0.42	0.29	0.40	0.42	0.29	0.29	0.24	0.03	0.41	0.35	0.25	0.55	0.58	0.12	0.33	0.24	0.21	0.31

CARP-CHOC-SECRET NUMBER

	AH7					AH8					FM2					FM3				
	Choc	Carp1-1	Carp1-2	Sys1-1	Sys1-3	Choc	Carp1-1	Carp1-2	Sys1-1	Sys1-3	Choc	Carp1-1	Carp1-2	Sys1-1	Sys1-3	Choc	Carp1-1	Carp1-2	Sys1-1	Sys1-3
	19	16	16	16	16	17	19	19	19	19	16	17	17	17	17	17	25	25	25	25
OKEQT	0.05	0.19	0.13	0.06	0.11	0.47	0.79	0.47	0.79	0.82	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.04	0.04	0.00
OKCALC	0.74	0.50	0.31	0.00	0.00	0.18	0.11	0.05	0.00	0.00	0.13	0.00	0.00	0.00	0.06	0.29	0.56	0.40	0.12	0.06
OKT&E	0.00	0.00	0.00	0.06	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.06	0.06	0.06	0.00	0.00	0.08	0.08	0.20	0.00
WEQT	0.00	0.00	0.00	0.13	0.42	0.12	0.05	0.42	0.16	0.18	0.06	0.00	0.00	0.00	0.00	0.00	0.00	0.04	0.04	0.06
WCALC	0.16	0.31	0.44	0.19	0.11	0.12	0.05	0.05	0.00	0.00	0.44	0.65	0.71	0.41	0.38	0.47	0.24	0.24	0.12	0.41
NATT	0.05	0.00	0.13	0.56	0.37	0.12	0.00	0.00	0.05	0.00	0.38	0.29	0.24	0.53	0.56	0.24	0.12	0.20	0.48	0.47
OK	0.79	0.69	0.44	0.12	0.11	0.65	0.90	0.52	0.79	0.82	0.13	0.06	0.06	0.06	0.06	0.29	0.64	0.52	0.36	0.06
WRONG	0.16	0.31	0.44	0.32	0.53	0.24	0.10	0.47	0.16	0.18	0.50	0.65	0.71	0.41	0.38	0.47	0.24	0.28	0.16	0.47
NATT	0.05	0.00	0.13	0.56	0.37	0.12	0.00	0.00	0.05	0.00	0.38	0.29	0.24	0.53	0.56	0.24	0.12	0.20	0.48	0.47

BUCKETS

	AH7			AH8			FM2			FM3		
	Buckets	**Sec+**	**Sec-**	**Buckets**	**Sec+**	**Sec-**	**Buckets**	**Sec+**	**Sec-**	**Buckets**	**Sec+**	**Sec-**
	21	21	21	17	17	17	20	20	20	24	24	24
OKEQT	0.05	0.24	0.10	0.29	1.00	0.71	0.00	0.05	0.05	0.00	0.00	0.00
OKCALC	0.90	0.29	0.05	0.59	0.00	0.00	0.60	0.45	0.10	0.88	0.50	0.17
OKT&E	0.00	0.00	0.00	0.06	0.00	0.00	0.00	0.05	0.00	0.00	0.17	0.00
WEQT		0.14	0.48		0.00	0.29		0.00	0.00		0.04	0.00
WCALC		0.24	0.19		0.00	0.00		0.15	0.40		0.13	0.71
NATT	0.00	0.10	0.19	0.06	0.00	0.00	0.20	0.30	0.45	0.04	0.17	0.13
OK	0.95	0.52	0.14	0.94	1.00	0.71	0.60	0.55	0.15	0.88	0.67	0.17
WRONG	0.05	0.38	0.67	0.00	0.00	0.29	0.20	0.15	0.40	0.08	0.17	0.71
NATT	0.00	0.10	0.19	0.06	0.00	0.00	0.20	0.30	0.45	0.04	0.17	0.13

PATTERN-SALESPERSON-SECRET NUMBER

	AH7	AH7	AH7	AH8	AH8	AH8	FM2	FM2	FM2	FM3	FM3	FM3
	Pattern	Salesp.	SecNo	Pattern	Salesp.	SecNo	Pattern	Salesp.	SecNo	Pattern	Salesp.	SecNo
	16	16	16	19	19	19	17	17	17	25	25	25
OKEQT	0.31	0.00	0.38	0.58	0.16	0.79	0.00	0.00	0.00	0.00	0.00	0.04
OKCALC	0.06	0.75	0.19	0.00	0.79	0.11	0.18	0.65	0.00	0.60	0.84	0.56
OKT&E	0.06	0.00	0.00	0.05	0.00	0.00	0.00	0.12	0.06	0.04	0.04	0.04
WEQT	0.13	0.00	0.19	0.11	0.05	0.11	0.00	0.00	0.00	0.00	0.00	0.00
WCALC	0.43	0.19	0.19	0.21	0.00	0.00	0.53	0.12	0.24	0.32	0.04	0.16
NATT	0.00	0.06	0.06	0.05	0.00	0.00	0.29	0.12	0.71	0.04	0.08	0.20
OK	0.44	0.75	0.56	0.63	0.95	0.89	0.18	0.76	0.06	0.64	0.88	0.64
WRONG	0.56	0.19	0.38	0.32	0.05	0.11	0.53	0.12	0.24	0.32	0.04	0.16
NATT	0.00	0.06	0.06	0.05	0.00	0.00	0.29	0.12	0.71	0.04	0.08	0.20

Annex E

Overall facility levels for all problems in the main experimental study

Overall facility levels

Questions		Location in test papers	% of correct answers
Sandwiches			**12**
Driving [2.7]	✓	**A1**	**16**
Seesaw [4x]	✓	**A2**	**22**
Slope B		**A2**	**24**
x+3y, x-3y	✓	**C2**	**26**
120-13n=315	✓	**A1**	**27**
Tickets [2.7]	✓	**B1**	**30**
Seesaw [11-5]	✓	**B2**	**32**
Sale [x, 4x]	✓	**A1**	**33**
x+y, x-y	✓	**B1**	**35**
181-12n=128-7n	✓	**B1**	**37**
Sale [11-5]	✓	**C2**	**39**
Clothes combin.		**C1**	**39**
Carp [1-2]	✓	**B1**	**40**
Speed B		**C2**	**46**
Choc [a+3b, a-3b]	✓	**C1**	**48**
Pattern	✓	**B1**	**49**
Slope A		**A2**	**50**
TV [commutativ.]		**A2**	**50**
20-(-10)		**A1**	**56**
6n+165=63	✓	**B2**	**56**
Speed A		**C2**	**57**
Carp [1-1]	✓	**B2**	**58**
Driving [4]	✓	**B2**	**64**
181-12n=97	✓	**A2**	**67**
Tickets [4]	✓	**A2**	**71**
Oranges B		**A1**	**78**
25-37		**A1**	**83**
Salesperson	✓	**B2**	**84**
Buckets	✓	**A1**	**84**
Oranges A		**A1**	**88**
Candies		**A2**	**94**

(Not all locations provided; marked items are analaysed in the dissertation)

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