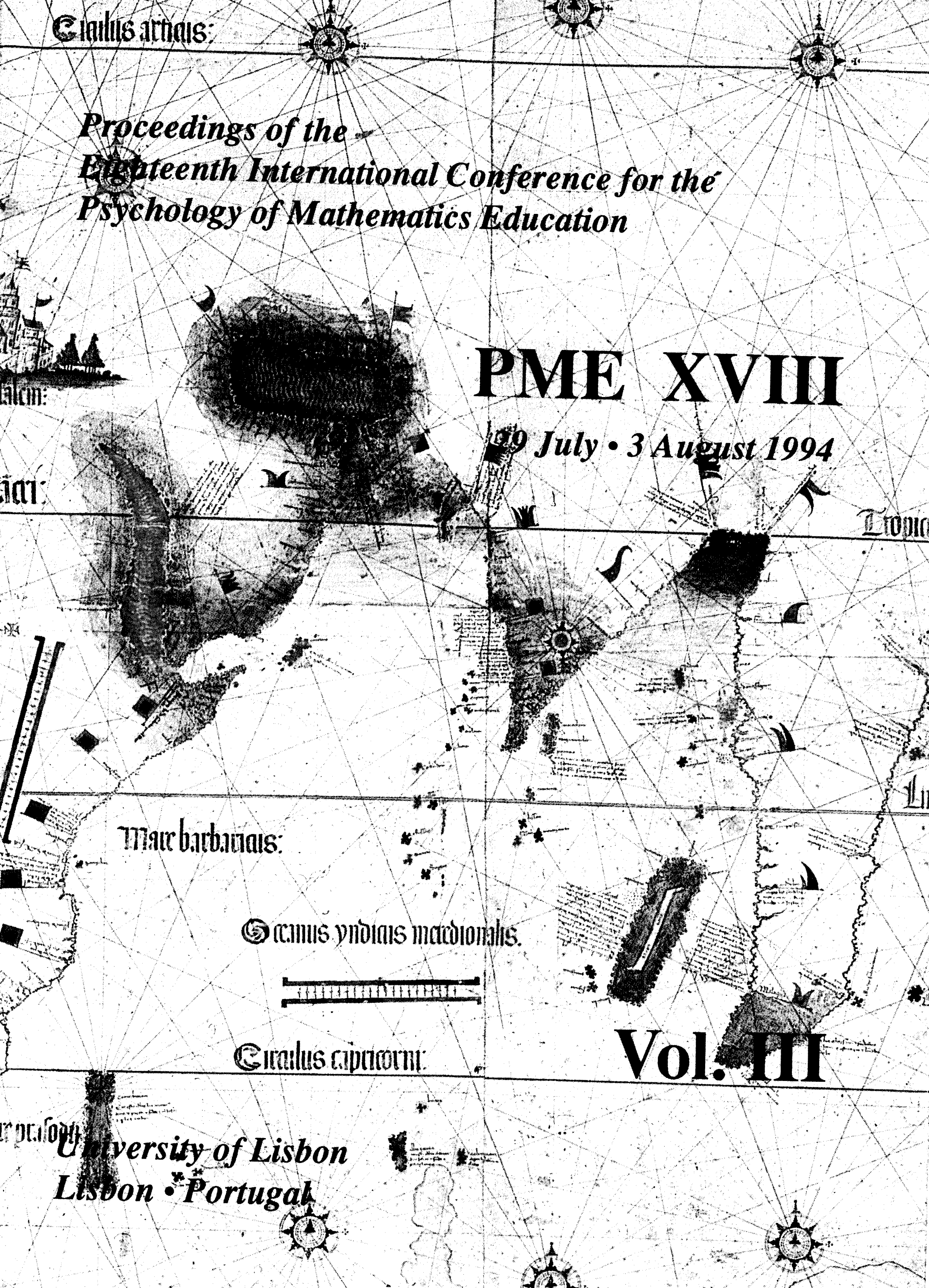

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Eliciting the meanings for algebra produced by students: knowledge, justification and Semantic Fields

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Abstract

For the past six years we have been engaged in developing a theoretical framework which accounts for meaning in mathematics, the Theoretical Model of the Semantic Fields. In this paper we discuss part of a 12-lesson long study with Brazilian sixth-graders (11-12 years-old), which is based on our theoretical framework. The central objectives of this paper are: (i) to argue for the importance of the theoretical construct ***Semantic Field*** *in the study of pupils'* ***knowledge****; (ii) to present and discuss the distinction between* ***solution-driven*** *activities and* ***justification-driven*** *activities.*

For the past six years we have been engaged in developing a theoretical framework which accounts for meaning in mathematics, in particular in algebra, including a characterisation of *algebraic thinking* (Lins, 1990, 1992a & b, 1993). As a result we have produced *The Theoretical Model of the Semantic Fields* [**TMSF**] (Lins, 1994). We acknowledge that the term *semantic field* has been used by other authors, mainly linguists (see, for example, Grandy, 1987), but also by mathematical educators (see, for example, Boero, 1992). Nevertheless, our conception of a *semantic field* must not be confounded with any of those, as it arises from an epistemological approach which is essentially distinct from the ones supporting those two conceptions—which are, by the way, also distinct from one another.

In this paper we discuss part of a 12-lesson long study with an intact class of Brazilian sixth-graders (11-12 years-old). Currently, a fully-fledged project is being carried out on the lines of the study here analysed[2]. The central objectives of this paper are: (i) to argue for the importance of the theoretical construct *Semantic Field* in the study of pupils' *knowledge*; (ii) to present and discuss the distinction between *solution-driven* activities and *justification-driven* activities.

The central research question in the study was the nature of an epistemological obstacle in relation to the "manipulation of the unknown," suggested by Filloy and colleagues (Gallardo & Rojano, 1987). It was our hypothesis, already supported by evidence from an earlier pilot study (unpublished), from an extensive study of pupils' solutions to "algebra problems" and from a historical study of algebra and of algebraic thinking (Lins, 1992a), that those difficulties were directly linked to the ways in which students produce meaning for the equations proposed to them. The teaching experiment, part of which is discussed in this paper

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aimed at showing that it is possible to develop a teaching approach which avoids difficulties with the "manipulation of the unknown," by producing a working context where that activity may become "senseful," and understood as one way—among others—of producing meaning for equations and their manipulation.

Theoretical Background

The theoretical support for both the development of the activities and the analysis of the results, is drawn from two sources: the *Theoretical Model of the Semantic Fields* [**TMSF**], and the ideas of Vygotsky, particularly in its influence on the work of V.V. Davydov.

At the heart of the **TMSF** is a particular conception of knowledge: *knowledge* is a pair formed by a *statement-belief*—that is, a *belief* which is stated—together with a *justification* for it[3]; for instance, one might say that in relation to the equation 3x+10=100 "one may take 10 from each side" (*statement-belief*) , with the *justification* that "it is as in a scale-balance." Such *justification* does not, of course, apply in the case of the equation 3x+100=10, but this does not imply that the *satetement-belief* could not be held by the same subject in relation to the second equation; another *justification* would have to be available, in which case a different *knowledge* would be produced.[4] Such mechanism has been shown to be of great relevance in the study of pupils' understanding of "algebra problems" (Lins, 1992a, 1993).

A second key concept in the **TMSF**, *meaning* is understood as the relationship between the *statement-belief* and the *justification* in a given *knowledge*, the full "being together" of the two elements in a *knowledge*. To say that a piece of mathematics is *meaningful* to a person, is to say that the person possess some *knowledge* about that piece of mathematics. A lack of understanding must, then, be seen as a lack of *meaning*. But if it is immediately possible to relate positively *meaning* and *knowledge*, the mechanism which allows relating the lack of *meaning* to the non-realisation of *knowledge* requires some further elaboration. The third key concept of the **TMSF**, that of *Semantic Field*, provides what is required.

A *Semantic Field* is *a mode of producing meaning*. We can speak, for example, of producing *meaning* for the equation 3x+10=100 within a *Semantic Field of a scale-balance*, or within the *Semantic Field of algebraic thinking* (see Lins, 1992a), or within a *Semantic Field of whole and parts*. But within the first or the last of those *Semantic Fields*, it is not possible to produce *meaning* for the equation 3x+100=10. A *Semantic Field* corresponds to possibilities of producing *justifications*, and, thus, of enunciating *statement-beliefs*. The same *statement-belief* may be justified within different *Semantic Fields*, but to each *justification* corresponds different *knowledges*,

[3] Although not originally derived from it, our conception bears a similarity with the classical definition of knowledge, $K_p=B_p$, p , J_p ("justified true belief"; see Everson, 1990); a detailed discussion of that similarity is found in Lins (1994). As to any objection in relation to the fact that *knowledge* must have been stated at least once, the reader is referred to Ayer (1986).

[4] Explicitly stated: the *knowledge* constituted by the pair (one may take the same from both sides, it works like a scale-balance) is different from the *knowledge* constituted by the pair (one may take the same from both sides, it is a property of numerical equalities).

A brief example might serve to show in what sense the concept of a *Semantic Field* throws light in the process of *knowledge* production. In relation to the equation 3x+10=100, a teacher and a student might agree on the *statement-belief* "we can take 10 from each side," although the teacher has a *justification* produced in terms of properties of the equality in relation to the arithmetical operations, while the student has a *justification* produced in terms of a scale-balance; there are *distinct knowledges*. It should not come as surprise—although so many times similar situations do, and, interestingly enough, also for researchers—that when presented with the equation 3x+100=10, and even being able to deal correctly with negative numbers, the student will say "it doesn't make sense." (see, for example, chapter 4 of Lins, 1992a). The false paradox arises when we mistakenly assume that the student should naturally "apply" to the second equation the *statement-belief* which had been enunciated in relation to the first equation; but *knowledge* is an irreducible composition of a *statement-belief* and a *justification*.

But what the concept of *Semantic Field* also indicates, is that while there is nothing in the equations themselves which can be linked with the production of *meaning*, the same is true of any environment or context, no matter how tempting it might be to say the opposite.[5] In fact, "real objects" are in themselves as "semantically empty" as any x's and y's can be. It is true, however, that *culturally* one situation will probably be more strongly associated with some *Semantic Fields* than with others, as is the case of situations involving money[6].

Although central in the model, the brief discussion of those three concepts—*knowledge*, *meaning* and *Semantic Field*—provides only the elements which are essential in the context of this paper; for a full presentation and discussion of the *Theoretical Model of the Semantic Fields*, the reader is reffered to Lins (1994).

Theoretical support to this study also comes, as we have said, from the work of V.V. Davydov, which is, on its turn, based on ideas from Vygotsky. Davydov has done intensive research on the teaching of mathematics for young children. In one of those experiments (Davydov, 1962) he started from modelling simple situations with whole-part models, and from there moved to exploring the manipulation of quantitative relationships. The use of literal notation was introduced rapidly and without trouble, becoming *a valuable and adequate tool in that context*.

We understand that the importance of those studies is twofold. First, they point out to the ways in which the use of symbols may produce a shift from the solution of problems to the investigation of methods of solution. Second, by starting from the manipulation of whole-part relationships, as a support, and then moving to the manipulation of the expressions themselves, the work of Davydov suggests a fruitful but not fully explored vein. In many respects it may be said that the passage from the "tanks" to the manipulation of expressions is not really different from approaches using "concrete material" or "contextualised situations."

5For instance, it is not the case that when dealing with a scale-balance one will necessarily operate within a *Semantic Field of a scale-balance*; some people will, instead, operate within the *Semantic Field of algebraic thinking*.

6It is hard to believe that even Paul Erdös would set and solve an equation to calculate the change in the market.

But there is a distinctive feature in Davydov's approach, namely that the whole-part model is used to *generate* the expressions which are to be manipulated, and not to illustrate the rules of manipulation; what is lacking, however, is the understanding that there are two *modes of producing meaning* in play, and that this situation should be explicitly addressed by teaching.

The combined use—in Davydov's work—of symbols interpreted within a familiar *Semantic Field*, within which the *logic of the operations* is sufficiently clear, together with the intention of systematising that *logic of the operations* into principles which would guide the manipulation of the expressions, naturally leads away from the traditional approach of achieving that objective through a generalisation of arithmetic. When Freudenthal (1974) says that "...generality is not always achieved through generalisation," he is in fact pointing out to the need of introducing a kind of activity in which generality is at the starting point, it is not just a target. To those activities we will call *justification-driven* activities, and they will be naturally opposed to *solution-driven* activities.

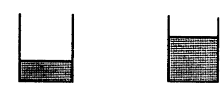
From Davydov's work, then, we borrow those two aspects: (i) operating within a familiar *Semantic Field* as a way of generating *meaningful* quantitative relationships in the form of expressions; and, (ii) the implicit distinction between *solution-driven* and *justification-driven* activities.

The conditions of the study

The study was carried out in 1990, at the Escola de Aplicação, a school set as part of the School of Education of the University of São Paulo. The activities were discussed with the class teacher, in order to guarantee that they would effectively contribute to the already planned teaching, and that they would not be seen by the students as mere "extras." Solving equations and using equations to solve problems were part of the program, and the only required change was in the planned schedule for the lessons. Students were told, however, that those lessons were part of an experimental teaching program. The researcher participated regularly in the lessons, sometimes in the role of a teacher. Students' work has been preserved in photocopies of their notebooks.

Classroom activity

The first activity proposed was based on a diagram given to the students:



With 9 more buckets, the tank on the left will be full; with 5 more buckets, the tank on the right will be full.

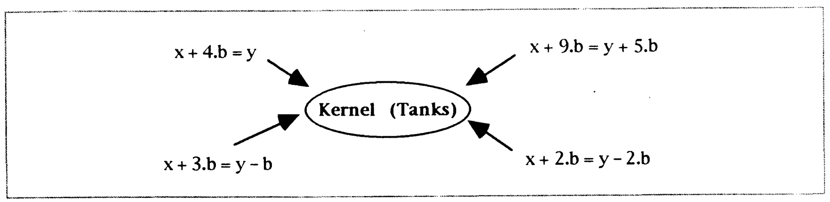
What can we say about this tank situation?

Students were then encouraged to produce, in small groups, expressions which could be shown to correspond to the situation, together with a *justification* for the adequacy of each expression produced. The use of "arithmetical" notation (using the signs for the arithmetical operations) was directly suggested by the teacher, and there was some negotiation as to the letters to be used. The water on the left-hand tank was notated **x**, and the water on the right-hand tank notated **y**, while a bucket was **b**.[7]

Some of the expressions generated, with their justifications, were:

Expressions	***Justifications* for adequacy**
x + 9.**b** = **y** + 5.**b**	"this phrase is correct because the two buckets [sic] will be a whole"
x + 4.**b** = **y**	"if I add 4 buckets to the tank on the left, they will have the same amount"
x + 2.**b** = **y** – 2.**b**	"**x** + 2 buckets will fill the tank, with 7 buckets missing [sic]. And in **y** there are 5 buckets missing, and if we do -2 becomes -7"
x + 3.**b** = **y** – **b**	"6 buckets will be missing on **x**, and on **y** 5 are missing; if still another 1 is missing, also 6 will be missing"

As those four examples indicate, the validation of each expression, ie, the production of *justifications*, was done by referring back to a kernel—the "tank situation." The students were operating within a *nucleated Semantic Field of a tanks situation*.

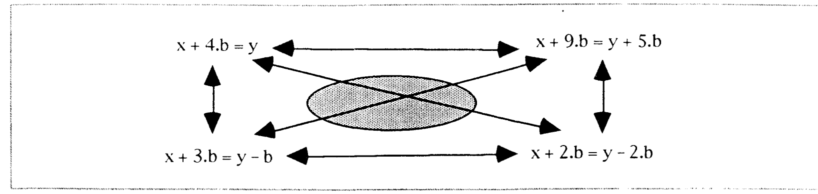


The next step was to propose another approach to the *justification* of the adequacy of new expressions. We asked the students to find a transformation of the expression **x**+9**b**=**y**+5**b** which would lead to the expression **x**+4**b**=**y**. The importance of having passed through the first phase—generating the expressions—is that there were now, available to the new activity, a large set of expressions to be manipulated, and those expressions were *meaningful*.The students had no difficulty in producing transformations such as "take 5 buckets from each side."

For each new expression generated from then on, the students were required to produce both a *justification* of its adequacy in relation to the kernel, and another *justification* of its

[7]In Portuguese "buckets" is "baldes."

adequacy from the application of a transformation rule to a previously established expression. Within the new *mode of producing meaning*, expressions are directly linked:



The use of the transformation rules produced expressions which could not be easily made sense of within the *Semantic Field of the tanks*, as, for instance, 2**x**+8**b**=2**y**, imposing the discussion of the differences between the two *modes of producing meaning*. The distinction was made even sharper when expressions were generated which the students could not be sure whether they made sense at all within the *Semantic Field of the tanks*, as in the case of **x**-30**b**=**y**–34**b**, once one could not be sure of the possibility of taking 30 buckets of water from **x**.

Having established some degree of independence of those specific expressions in relation to the kernel, we could move to the manipulation of expressions which had not been generated within some *nucleated Semantic Field*, as, for example, transforming the expression 3**x**–4**a**=2**y**. This part of the work was always carried out with a target in mind, for instance, transforming that expression in a way to obtain another expression, of the form 4**a**=... .[8] The fact that the students could correctly deal with this kind of task, suggests that the exit in the previous part of the teaching was not due to the support offered by the "context" of the tanks. Rather, once the direct manipulation of expressions had become a "senseful" activity, not only the technical difficulties did not occur, but, also, the students began to bring into play methods produced in arithmetic (for example, simplifying simple rational expressions).

Discussion and conclusions

The aspect which will be discussed here, of the teaching experiment partly presented in the previous section, is the role played by the theoretical framework in the design of the activities and in the analysis of the results. Any lenghty examination of the learning outcome of the approach we propose has to be preceeded by that discussion; the presentation and discussion of a teaching approach based on the **TMSF** will be found somewhere else in the near future.

The theoretical construct *Semantic Field* was at the heart of the process of designing the activities, pointing out to the need of having pupils to present *justifications* for the correctness

[8]In choosing this format for the activity, we had in mind the introduction of a strongly analytical perception of the expressions (Cf. Lins, 1993)

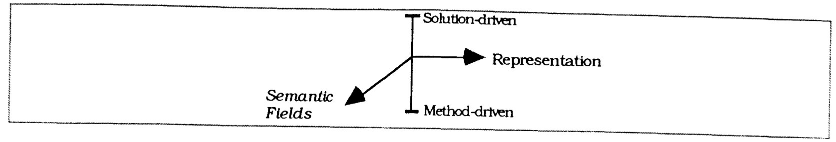
of the expressions (*statement-beliefs*) produced. The importance of those explicit *justifications* is twofold. First, they supported the introduction of the direct manipulation of expressions as one way—among others—of making sense of producing new expressions; in fact, the manipulation of expressions is seen, in the context of the activities proposed, as the production of new, adequate, expressions.[9] Second, in order to focus sharply on the production of *justifications*, we were led to design activities where the possibility of producing particular numerical solutions was denied: instead of "find a solution" activities, we proposed "make sense" activities. Bruner (1988) had already pointed out a possible, and very interesting, paralell between the notion of *given* and *new* tokens in speech—introduced by linguists—and the behaviour of subjects prompted to "think aloud" while solving problems. Bruner observes that those subjects produce a speech (which is likely to be a spoken version of the inner speech) from which the *given* is very much supressed. From the point of view of the **TMSF**, *justifications* certainly belong to the class of the *given*, as they must be accepted before being able to provide an anchor to *new statement-beliefs*. The format adopted in our activities, led the students to deal with both the *new* and with the *given*; as a consequence, they were not working only with solving problems, but also working on producing and enriching—and internalising—new *Semantic Fields*, ie, new *modes of producing meaning*. In Lins (1994) we present a full discussion of the role of *internal* and *external interlocutors* in the process of developing *Semantic Fields* (*knowledge* production).

The fact that our students did not have any substantial difficulties in dealing with literal expressions suggests, in the light of the **TMSF**, that this process is directly linked to the ways in which *meaning* is produced for those expressions. As we had already indicated (Lins, 1993), the "analytical behaviour," dealing with the unknown as if it were "known," is subordinated to particular characteristics of the *Semantic Field* within which the student is operating.

The two key aspects of the dynamics of the teaching approach adopted, are what we call *vertical* and *horizontal developments*. The former consists in the production of new *statement-beliefs* within a given *Semantic Field*, while the latter consists in the reinterpretation of "old" *statement-beliefs* within another *Semantic Field*. *Vertical development* enriches *modes of producing meaning*; *horizontal development* enriches the overall capacity of a *system of knowledge* to produce new *knowledge*, but it also enriches the global *meaning* of *statement-beliefs*.

Based on the theoretical framework, and on the overall results of the teaching experiment, we suggest that the design, conduction and analysis of classroom activity should be considered on a three-component system:

[9]In the process of "properly" solving equations, each transformation is seen as specialised, in the sense that it is almost necessary; although in many teaching approaches one finds the requirement of adding "justifications" to each "step"—eg, "do the same to both sides"—those transformations are dealt with from a very narrow perspective, and as a consequence, the idea of using those transformations to articulate expressions in a way to express more than initially available—for instance, manipulating an expression to show that the number of black tiles on a pattern is always even—is not developed, in the sense of it not being a legitimate strategy.



This system should not be seen as a mere "change of basis" in relation to other systems. Although the "concrete-abstract" distinction can be formally interpreted in terms of the three components of our system, such exercise is of no interest. The **TMSF** aims at replacing such traditional polarities with a more flexible and precise framework. We think that research conducted within the framework of the **TMSF** should be concerned with producing a distinct approach to teaching: what to teach, how to teach, rather than with solving learning difficulties which are—more likely than not—produced precisely by the epistemological assumptions underlying those teaching approaches—eg, that there is a "path" from "concrete" to "abstract," and even the assumption that those two categories correspond to qualitatively distinct kinds of *knowledge*.

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