# The need for emphasizing global arithmetical and algebraic sense and meaning

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## 1. Introduction .

There have been great and deep changes in the world during the past 30 years or so, and that is well known. Schools, however working on catching up with those changes, are still struggling with producing fresh ways to approach the problem of finding itself a way of being truly useful, of helping people to be better equipped to live in such a changed-and changing-world. Despite the introduction of new technologies and of strategies such as problem-solving, school still is, to a considerable extent, bound to a traditional curricular organisation, in the specific case of mathematical education, one based on the "arithmetic, algebra, geometry" triad.

Research should be leading the necessary process of change, but-also to a considerable extent- it has been mainly reproducing that traditional division. The evidence for it, is that research focus is strongly produced in terms of *contents* - "naturally" leading again to "arithmetic", "algebra", "geometry"- and not in terms of some other category as it occurred in Natural Sciences Education Research. It is not obvious that the traditional triad does not offers the opportunities for efficient change, although there is evidence for that.

## 2. Trends for unification.

Academic Mathematics has, for quite a while, argued that viewing arithmetic, algebra and geometry as separate things is, to say the least, a mistake. From the time of Descartes, a process started towards some kind of "unification", a process which reachedits maturity with the formal version of Mathematics offered by Bourbaki. Viewed from that point of view, plane curves "are" equations, one can measure the distance between functions, and quaternions -which can be neither visualised geometrically, nor have some of the basic properties we'd expect from numbers-are numbers.

Formalism has not been the only one to advocate for unification, but being the main trend in academic Mathematics, it's from it that the unification view reached school. That view had some influence on school mathematics, mainly through the so-called Modern Maths movement, but as the latter was abandoned, so was much of the unification view, and school mathmeatics tended to fall back on the traditional triad. It has to be said that the unification view never had the time to reach school mathmematics in full, ie, many teachers never really went into it, and as textbooks stopped conveying it, there was little memory left of it to them.

Coming back to research, it is interesting to consider the extent to which researchers are identified with "content": "he (or she)'s an expert on fractions"; she does great work on geometry"; there comes the algebra man!"; it sounds very much as if we were talking of working mathematicians. This is not something against specialisation: it just points out to *what we are specialising on* . There are, of course, those who produce cross-boundaries research, but they are not many, and for good reasons, as it takes a good number of years of concentrated work before we really feel within one of those areas. Although broad frameworks, such as Vigotsky's and Piaget's do exist, and still have a strong background influence, there has also been a somewhat recent trend towards "local theories" that meaning theoretical constructs aimed at explaining specific sets of data coming from experimental work; such a trend reinforces, of course, whatever underlying research organisation is in place, in our case, a content-based one. We do not intend to go any further into analysing present trends in research and classroom practice in mathematical education. We will, instead, argue for a different view on the relationship of algebra and arithmetic, one that brings together a number of points which mathematical education has been trying to adress in recent past.

## 3. Necessary loves and divorces. Is there a winner?

We start by considering the relationship between algebra and arithmetic. On the one hand, there is a clear agreement that there are very different, a view which emerges in notions such as epistemological break point (Chevallard 1991) didactical cut (Filloy & Rojano) and in a number of studies dedicated to examining the *transition* from arithmetic to algebra. Algebra is "abstract", while arithmetic is "concrete"; algebra is structural, while arithmetic is procedural. On the other hand, algebra is also seen as emerging from arithmetic, this view being supported by a historical account of their development, or simply by the notion -still very strong- that algebra is generalised arithmetic. We should also consider that there is a strong school tradition, against which any new view on that subject is to be opposed: for many centuries, arithmetic has been taught before algebra, and most people have at least some success in it, while algebra, even coming much later in school, is strongly related.



A number of studies have pointed out that many of the difficulties in algebra are related to the learning of arithmetic. Some of them suggest that it is the lack of a sufficiently built knowledge of arithmetic which is responsible for difficulties in algebra, while other studies suggest- quite on the contrary - that it is precisely the consolidation of notions typical of arithmetic, which constitute an obstacle. An example of the former would be an insufficient mastery -within the domain of specific numbers -of the distributivity of multiplication over addition; an example of the former would be the meaning of equal sign as an indication that a calculation must be performed. It is certainly a paradoxic scene, but there is a common feature to both sets of findings: arithmetic comes first.

## 4. Within and outside school Sense making .

Let's stop thinking for a minute in terms of pre-requisites, and ask a much more fundamental question : what could be the *sense* [1] of learning arithmetic and algebra ? There is, of course, the long-standing answer, "it's necessary in many professions", or less cynical one "it is a necessary step in going through school life". But that is not the sort of answer we are looking for. A more useful one is to say that algebra and arithmetic should be part of *organising* people's activities, and that learning them should be part of broadening the possibilities for that organisation. A first consequence of such a statement, is that the learning of arithmetic and algebra should *serve* the overall life of people, and not be served by it.

The learning of algebra and arithmetic is served by overall life, precisely when one suggests that "reallity" be brought into the classroom to function as a support for the learning of algebra and of arithmetic, be it in the form of "applied problems", or "real-life problems", or the form of concrete objects" (scale-balances, for instance); there is also the idea of "building-up from informal methods". The general idea is that "concrete" settings would provide the basis for the learning of "abstract" school mathematics.

The learning of algebra and arithmetic serves overall life when it answers to organisational needs which emerge from actual demands of life. Simply put as it is, all we have is the usual suggestion that learning be "really useful" in a way that motivates students. But there is a catch. Whenever we say that we

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[1] Sense was taken as a synonim of set of feelings (Sowder 1992) or behaviours (Fey 1990) in which the context plays an important role. In the Arcavi's description (Arcavi 1992) sense is characterised by friendly approach, ability for readigness, operatory flexible abilities and analysis of a diversity of contexts. In our perspective we follow a more simple semiotic view (see section 6 ).

should bring "real-life" into the classroom, we are implicitely saying that classrooms are not "real". But what could that mean ?

It could mean, for instance, that going to school is like putting life in brackets for a while, doing something which vanishes as soon as one leaves school, both phisically -everyday- and formally- at the end of schooling. This is the feeling of many, if not must , students, and fits most perfectely into the "let'sbring life to schools" moto. We could classify it as a content-driven view: "why do I have to study equations?" Because schools almost always organise things around contents. Not only a question thus formulate is adequate, but it is also adequately answered with the obvious "it will be useful later". There is however, another possible meaning for school being taken as not real. It could be that the key difference is not related to contents, but to what is considered legitimate as ways of thinking. We may start considering arithmetics within and outside school.

Outside school, a good approximation is frequently all one needs, while within school approximations are acceptable only in specific cases - rational approximations of irrational numbers, for instance. To think in terms of approximation is legitimated differently within and outside school.

It is not properly a matter of content, as in both cases one is interested on and actually doing calculations. But the fact that approximations are allowed-encouraged, in fact -outside school, leads to a peculiar situation, one which does not interest much to school, namely, that because in different situationsthe accuracy requirements are different, different strategies for calculations are bound to emerg. In dealing with money, for instance, it is not necessary to think in terms of the decimal dot, as the currency itself marks what is to be added to that and in which ways, while in dealing with measurements, for instance,it is possible for most practical purposes, to use a unit which makes everything into whole numbers.

Within school, however, everything is treated as if no external reference coul ever be of help in suggesting particular strategies. If I have to multiply 3.1 by 2.9 there is nothing to help me to decide whether 9 would be a good answer or not, and I am left with doing the precise calculation. School procedures *must* be general, precisely because they are not subordinated to any particular non-mathematical activity, ie there is no non-mathematical meaning involved - in theory... There is nothing intrinsecally wrong with that, and "scientific" mathematics has been doing it-quite succesfully- for a long while. The point is simply that most people do not, and never will, think like that. Maybe we could rewrite the complaints of students as,"why do I have to be so precise?"



From this point of view, the "lack of reality" within mathematics classrooms is similar to the one felt by students complaining that a given school does not allow male students to use long hair, a matter of legitimacy, not of content.

We think it is worth examining a simple example from that point of view. Let's consider a teacher who, trying to motivate students by bringing "real life" into her/his classroom, decides to promote lessons around the theme "kites". Children , like kites, and like making them. The teacher's plan is to explore geometry -forms and measurement- and prepares to get the students into talking of rectangle triangles, simmetry, *precise* measurements and *areas* . It all seems just too reasonable, but we must ask the question: what difference would and make *for tha actual making of and playing with kites ?* The answer is, of course, none. What would the children make of all that,then ? Well, they may simply feel that, since it has been proposed by the teacher, they'd better keep doing it; some of them may come to believe that unless the right angles are perfect, and the measures precise, the kite is not worth it- and those are strong candidates to future mathematicians.[2] What seems weird in such situation is not the idea that children would rather talk of kites than of "abstract" shapes, but the fact that it seems absolutely natural, for the teacher, that making kites naturally "embodies" those mathematical notions. What sense is there in measuring the area of a kite?

The key point is that although there might be a kite both within and outside school, thinking is much different in any case, and values such a precision and quantification, so valued within school, may not be so important in making kites outside school. Within school simmetry leads to precise measurement, while outside school it leads to physical equilibrium.

Returning to arithmetic and algebra, the perspective we have provided so far suggests that we examine the thinking normally associated to them within and outside school. As to arithmetic, we have already said something.

As to algebra, the situation is somewhat different, simply because there is no visible algebra outside school. To say that it is in Engineering, Physics, Computing, is not convincing, because most people will not become engineers, physicists or computer scientists; it is certainly more adequate to consider those and similar professions as still "school" in a wider sense. On the other hand, school algebra embodies some "thinking values" which are foreign to life outside school; precision is one of them, as we have already

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[2] In his *Weeding and solving*, Hans Freudenthal mentions a similar situation concerning students who had had instruction on descriptive statistics, and mentions the "perversity" on the thinking of those who speak of their friends "taste for ice cream on the basis of fictional-statistical data about the whole population's taste.

pointed out. Also, although school algebra may be seen by experts as "calculating with letters" it is difficult to think in those terms outside school, were calculations are always made with the purpose of determining an actual, needed, result. This is not to say that people are not "intellectually" able to do it: it's just that there doesn't seem to be any sense in doing it.

Generality is another characteristic of algebra, not because it deals with "generalised numbers" -an uninteresting conception of algebra-, but because it is so centrally concerned with *method* . Strictly speaking, school algebra has fewer basic operations than arithmetic, as it does not have to distinguish between whole numbers and decimals, the rest being common to the two. In fact, we manipulate an expression such as "$23 \cdot (12- 5 \cdot 7)$ " by doing successive calculations and substitutions in order to evaluate it, and even we do, some specifical writing transformation such as $\begin{array}{r} 23 \\ \underline{\times\ 23} \end{array}$ in order to perform a calculation .

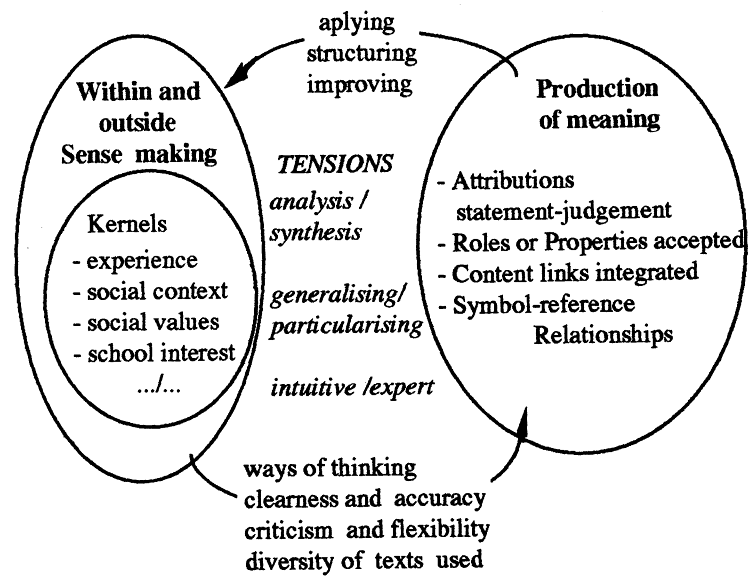
It is true, however, that expressions in arithmetic are much easier to deal with, because all that is required is the identification of the correct order of operations, and from there following a series of quite standard and "terminal" transformations. [3] Algebra, on the other hand, is characterised precisely for dealing with general transformations of expressions and, for that reason, it stresses the importance of clearly establishing relationships that model a problem, but it also offers -quite naturally-the tools to do that. But that structuring is precisely aimed at finding a way to produce -from the data and through the application of arithmetical operations. No matter the impression one may have, manipulating an equation does not solve the equation , it only offers "candidates" for solutions; the actual solution only comes when a "candidate" is shown to be a solution to the equation we are trying to solve, and that may be done both by substitution and checking and by some more general method. Technically speaking, a guess is a good as any other way to reach a solution. In other words, wile algebra is analysis aiming at synthesis, arithmetic is synthesis guided by analysis[4] . The key idea is that in helping

[3] The transformations in arithmetic are, in summary, those required to assemble the numbers so to allow the performance of an algorithm, and then substituting the result for the calculation which produced it. Those transformations are terminal in the sense that once their purpose is reached, they completely disappear from the structure of the expression.

[4] Synthetic and analytic are used here in the sense intended in ancient Greece, ie: Synthesis is the process by which one proceeds from what is already known, producing from there new truths; Analysis is the process in which one takes as known what is being asked, and proceeds from there, seeking for consequences, until



people to get better prepared to live, school should rely on a much closer between algebra and arithmetic. Such an account of the relationship between arithmetic and algebra moves us away from a content driven view, precisely because it becomes clear that the algebraic activity aims at the arithmetic activity, while the arithmetic activity is structured, organised, by the outcome of an algebraic activity. On another front, should also become open to ways of producing meaning which are not only the mathematical ones, both for algebra and arithmetic. That means giving up the idea of bringing reality into school, substituting that notion by the much wider one of making school a place in which meaning is produced -in many different ways- for algebra and arithmetic among other texts.




In a more general view, making sense and producing meaning are strongly related as we see in the figure. In fact, in a recursive understanding model of buiding knowledge, such a process is a continuous cycle in which school system try that sense and meaning is being continuously improved.

Next two sections are devoted to establish some specific examples of these relations in the case of algebra and arithmetic. It will justify the need for

something admittedly know is reached, at which point the process is "reverted" and made into Synthesis.

emphasizing the tensions and principles more than the mathematical topics itself. This educational perspective press for finding the already known relationships between arithmetic and algebra into a new perspective.

## 5. Producing meaning.

Arithmetic and algebraic activities are not to be seen as characterised through content, but through ways of thinking, ways of producing meaning. On the one hand, algebraic activity is characteristically analytical, while arithmetic is synthetic , and in that respect they are very different, on the other hand, they are both ways of organising aspects of human activity, and both deal with relationships involving numbers or sets, arithmetical operations and equalities or inequalities.

If the procedures of algebra are to be understood by students as being justified by the properties of numbers, the answer is "yes", and that is the "generalised arithmetic" view of algebra. But there is nothing in the characterisation we have provided which makes that a necessary fact. As much as there are research studies, showing that people produce meaning for numbers using a variety of assumptions -in most cases, assumptions closely linked to the situations in which those number appear- there are also studies showing that people produce meaning for expressions of algebra in the same way.

Numbers can be "number of money" or "number of things", as much as equations can be "scale-balance equations" or "undo-machine equations" among others. In such a characterisation, 3,4567 cannot be a "number of money", -12 cannot be a "number of things", "3x+100=10 cannot be a "scale-balance equation", and 4x+10=x+100 cannot be an "undo-machine equation.

The truth is, meaning for "number " develops from many sources and experiences, and many of those can as well provide meaning for "equation". A simple whole-part relationship might provide much insight, for instance, on the fact that a number can be decomposed as many different sums, but it can also provide much needed insight into the fact that,

$$a = b + c\ ,\ a - b = c\ ,\ \text{and}\ a - c = b$$

are each "valid consequences of" or "valid transformations from" or "equivalent to" the others, and there is enough evidence -for instance, from



the work of V.V.Davydov - to show that such insight can develop, even in very young children, directly from general situations[5] .

The logic of the operations is not built from numbers, but from whole and parts, and it provides both the basis for producing both knowledge from algebra and for developing a number sense which is much more refined that one starting from units, tens and hundreds. We may speak of a great variety of those kernels - whole-part, undo-machines , money, -in relation to which meaning is produced for both "number" and "equation" and that solves, from our point of view, the question of who'd come first: none, arithmetic and algebra develop *together* , and that is in agreement with what we have said before, about the impossibility of algebra or artihmetic being relevant without *some* of the other.

## 6. New tools introduce new senses ?

Here we consider briefly the comprehensive question of helping people to develop a "number sense". In fact, the sense of a proposition is not only reduced to a semmantic content, but also includes the judgment and value after their statement (Duval 1995). School has traditionally failed on that task, and on two counts. One of them is being closed to ways of thinking typical of life outside school mathematics, as we have pointed out. The other is by completely ignoring that "number" is not always what it is in school mathematics.

There are many situations in which "numbers" are not to be added or multiplied. They are not to be ordered. This is the case of telephone numbers. nevertheless as much as should be able to recognise units, tens and hundreds, and work with them, one should be able to recognise that the first digits of a telephone number indicate something different from the others. The idea of working with numerical code numbers of this sort is not new, but it means to insist that school has to open itself to other ways of producing meaning. Considering the use of numbers to grade gymnasts in the Olimpic Games could lead to a fruitful discussion of how order can be taken separately from calculating with numbers, but also to the fact that using numbers to that end does not lend the grading any further valua, and from there, maybe teachers would like discuss the grading of students in a maths test. Car plates is another example: why do they use numbers, even if no ordering is required ?

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5 Davydov uses, for example a situation in which there is a parking lot, with trucks and cars, all of the vehicles, to arrive at T+C=V, etc.

It may quickly become visible that meaning for number can be produced in different ways, and there is also a contribution for the learning of the "traditional"school topics : "if there are different ways of producing meaning for number, let's take a look at these..." That would help to legitimate school.

In these examples, people uses numbers in such a different ways, that students can observe and produce different meanings. But there is some individuals but sense is usually promoted by the situations. This tension is evident in the classroom.

Let's look at the availability of new ressources, such as calculators or computers. Do it suggest any new approach for making sense? Whenever it is proposed that their existence somewhat makes unnecessary the teaching of algorithms and procedures in algebra and arithmetic, someone comes with the "but what if we find ourselves without them and having to do a calculation or solve an equation?" As much as life became organised around new tools as machines, or motorised transport, life is becoming organised around calculators and computers. It promotes new challenges for education. In fact, calculators and computers can produce information in quantity and quality we never dreamed of. You ask a calculator to perform a simple calculation as 35 ÷ 6 which anyone knows comes up to around 6, and you get a mystical 5,83333333. What to do about it? Or... ask Mathematica to simplify a polynomial and you might get a surprise... While before much of school mathematics was about producing numbers, now it will have to shift to making sense of them.

## 7. Conclusion.

All these perspectives are not separated from the so called dialectic tool-object (Douady 1986). That perspective focused on the new role for representations when mathematical objects were introduced, bu now the focus is centered on a semmantic fields perspective by which having sense is strongly linked to the production of meaning itself. In our perspective, there is no previous "mathematical object" to be exploited, but a situation opened to the students' production.

What has been said so far stablishes, we think that:

(i) algebra needs not to be seen as generalised arithmetic

(ii) arithmetic needs not to precede algebra

(iii) meaning for arithmetic and algebra can be produced in relation to many different kernels.



More important, however, it points out to the key role of algebra with respect to the arithmetic, that of organising arithmetic activity, it is beyond doubt that when solving a problem involving quantities, a person has to establish relationships and draw conclusions which tell him/her what calculations to do in order to arrive at the required solution. It is also well documented by research that a major difficulty with such problems is identifying the relevant data and establishing the relevant relationships. Some of the most striking findings, however, suggest that many people who are "good" with algebra do no use it as a part of that organisational process; rather they see it just as another "content" - and that is how it is presented in school- and react quite naturally, asking, "why add another problem on top of the one in hand?" (see the work of Lesley Lee and David Wheeler).

This is not the place to explore the full consequences of the view we propose, and we will leave here the matter of a more interesting approach to the combined teaching and learning of algebra and arithmetic, suggesting that it be deply reconceptualised, particular with respect to early grades. The one key feature would be paying close attention to the roles of algebra and arithmetic relative to each other's, and always seen within the process of organising information, be it in solving a problem or in an investigative task.

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