Perspectives on School Algebra

Edited by

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APPROACHES TO ALGEBRA

In July 1991 the PME[1] working group 'Algebraic Processes and Structure' met in Italy for the first time. Our aim was to characterise the shifts that appear to be involved in developing an algebraic mode of thinking, with a particular focus on the role of symbolising in this development. For the next four years we met in the United States, Portugal, Japan and Brazil and discussed papers which had been circulated before these meetings. The gestation period for this book has been a long and productive one, resulting from ongoing discussion by members of a group who, although often disagreeing, were willing to share and debate their ideas. In this way we believe that we have developed our own thinking about teaching and learning algebra, as well as about the nature of algebraic thinking, although the earlier algebra research formed an important background to much of the research represented here (see for example Kieran, 1990).

In all chapters we find a concern with identifying the characteristics of what could be called an algebraic approach to solving a problem, together with a focus on the meanings the students construct/produce as they engage with mathematical problems, and not on the problems or the students' conceptualisations in isolation from the problem solving activity.

Within this picture, however, we find two distinct trends. In one group of chapters the main concern is related to what has been previously called 'informal' or 'spontaneous' approaches/meanings, and how those take part in the development/acquisition of an algebraic approach/algebraic meanings, whereas in another group the main concern is related to what an algebraic approach is in itself. The former could be identified as a *didactical* trend, while the latter could be identified as a *foundationalist* or *theoretical* trend. There is not, of course, a sharp divide here, as all the 'didactical' chapters deal, in one way or another, with some foundational issues — sometimes in the form of implicit or explicit assumptions, and all 'theo-

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retical' chapters deal, in one way or another, with issues about the teaching and learning of algebra. Nevertheless, these are useful categories which characterise the work presented here.

The interplay between these two trends was always a very rich one within the group, as it represented a balanced effort to: (a) clarify and deepen our understanding of the processes involved in algebraic and non-algebraic thinking; and, (b) to make sure that this improved interest was always informed by and linked to our educational endeavour. Those more closely concerned with teaching and learning in the classroom were faced with questions beyond the effectiveness of teaching approaches, while those more closely concerned with theoretical issues were faced with relevant and informing input coming from work with students; we had the exciting opportunity to work within the environment of a 'full table' with each didactical approach being confronted with different theoretical approaches, and with each theoretical approach being confronted with different didactical ones.

In any research concerned with algebraic education we should expect to find these two components, but our group had the opportunity to emulate them on a larger scale. In this sense we feel that this book should be read as a whole — heterogeneous as it may appear to be — and not as a collection of individual contributions.

Another useful way of characterising the chapters is by looking at the different horizons each one sets itself.

In one chapter we may find, for instance, a concern which falls almost completely within the domain of mathematics and of mathematical meanings, while another chapter allows for social and cultural issues to become of central interest. Also, different chapters may deal with so-called local theories, while others propose a perspective which starts from a broader view of the issues at hand.

This is not, of course, a simple matter of 'wide vs. narrow approach'. The key issue here is that in each case we have, effectively, the proposition of a perspective for research in mathematics education. Although the same could be said about almost every edited book in the field, the difference here is that we actually *lived* this diversity, and had to deal with it during our discussions. Each one of the controversial issues found its way to the spotlight, and that made us much more aware of them than is usually possible by reading the literature and attending conference sessions. Year after year, these differences would be revisited, and then either resolved, deepened or simply abandoned. At some point someone would say that history or epistemology was not relevant; some would say that our effort should be put into making available to teachers what had already been developed. We never reached a full consensus—as this book shows — but this is only to say that we were always engaged in an effort to reconceptualise the field, and in such situations difference is a much needed fuel.

Until some ten years ago, research and development related to algebraic education was strongly dominated by a concern with *notation*. There was a mostly implicit assumption that algebraic thinking could only happen in the presence of 'letters', and arithmetic was at best seen as part of 'pre-algebra.' Although acknowledging the power generated by the use of algebraic notation in many situations, and maintaining that one of the key objectives of school algebraic education is to get students to master its use, much of the research and development presented in this book takes into consideration an important issue: no matter how suggestively 'algebraic' a problem seems to be, it is not until the solver actually engages in its solution that the nature of the thinking involved comes to life. A related issue is that the signs of algebraic notation do not carry, in themselves, any meaning that can be apprehended by simply examining them.

Taken at face value, this may seem all too obvious. But what the working sessions of our group slowly revealed was that it has subtle yet powerful implications.

One of these is the crucial role played by teachers — as opposed to a previous focus on some 'natural' development of algebraic thinking. All chapters in this book deal, more or less explicitly, with the need for intervention if the students are to make/survive the cuts/shifts/reconceptualisations (from arithmetic, mainly) required for the development of algebraic thinking. Sometimes it comes in the form of a statement of how students should think with respect to given aspects of algebra — and, implicitly, that someone has to tell them this, otherwise they would not be making mistakes; sometimes it comes in the form of how other peoples thought in the past — together with a statement to the effect that this does not suggest any 'natural' route.

But there is another equally important side to this emphasis on the teacher's role, and one which adds, we think, to previous approaches. Not only will the teacher have to have ways to *intervene*—for instance, through sequences of well crafted problems and situations — but s/he will also have to be able to read what the students are saying/doing, and the outcome of this will affect the course of teaching. Maybe this is a good point to remember the powerful notion, highlighted in the work of Vygotsky, that any process brought into play will necessarily cause its own transformation.

Not that this has not been a tenet for good teachers for a long time, but it seems that finally it is more and more being *in-built* into development and in the research supporting it: teachers do not only need good material, they also need good 'reading' strategies.

Again, the balance between 'didactical' and 'theoretical' issues seems important. Didactical approaches in this book will, as usual, focus on the intervention side. However, the theoretical approaches put forward here focus mostly on the reading side, rather then on prescription. A balance is achieved, but not one between 'foundations of teaching', in the sense of 'how things are' and 'how teaching should proceed', and 'teaching' — the actual practice; there is, instead, a balance between teacher and student meanings, a balance between teacher and students *interventions*.

INTO THE BOOK

Maybe we should now take a look at individual chapters.

Chapter 2, by Luis Radford — 'The Historical Origins of Algebraic Thinking' — opens the book with an insightful examination of the origins of the notion of the 'unknown', pointing towards proportional reasoning. Radford examines material from the mathematics of Babylonia, ancient Egypt, ancient Greece, the Middle Ages and Renaissance Europe. The objective is not to trace a direct route, but rather to analyse similarities and differences. In his way, he looks at aspects both internal and external to mathematics, something which will be reflected in his final remarks: '...each algebra (Mesopotamian and Greek) was conceived deeply rooted in and shaped by the corresponding sociocultural settings. This point raises the question of the explicitness and the controlling of the social meanings that we ineluctably convey in the classroom through our discursive practices.' (page 34). In other words, there is an interplay between pragmatic needs and symbolic invention, and the pragmatic needs may well fall outside the internal needs of purely mathematical developments.

Chapter 3, 'The Production of Meaning of Algebra: a Perspective based on a theoretical model of Semantic Fields,' was written by Romulo Lins. It provides a reconstruction of the notions of knowledge and meaning, and analyses the consequences of this reconceptualisation for algebraic education. The perspective is clearly epistemological, the main question being 'what are those students saying?'. It is a 'theoretical' chapter, but the theoretical discussion leads to a classroom approach, based partly on a distinction between activities which are 'problem-driven' and those which are 'solution-driven'. The main arguments are centred on a notion of *text* which denies any intrinsic meaning to algebraic notation, looking instead at algebraic thinking as a way of producing meaning — among many others — and the implications for algebraic education deriving mainly from the need to establish the legitimacy of algebraic (mathematical) meanings and from the need to elicit the meanings students produce for a given (algebraic) text. The role of the teacher is crucial, but the broader notion of *interlocutor* is discussed.

Chapter 4 is 'A Model for analysing Algebraic Processes of Thinking'. Ferdinando Arzarello, Luciana Bazzini and Giampaolo Chiappini work within a horizon that includes both linguistics (Frege) and epistemology, providing an insightful look at what an algebraic expression represents (*denotes*) and that which it suggests or shows (*sense*); much analysis is directed towards the (negative) effect of collapsing

this distinction. It is certainly a theoretical chapter, but throwing light on classroom processes. The authors present a number of useful notions for analysing the process of meaning production/construction in the process of solving problems presented in algebraic notation, such as 'evaporation', 'condensation', 'algebra as a game of interpretation', and 'conceptual frame.' In all, there is as much a concern with mastering basic algebraic manipulation as with putting algebra to a more advanced, developed use.

David Kirshner's chapter 5 ('The structural Algebra Option Revisited') takes a different approach. His main argument is neither for a greater attention to student's approaches/meanings nor to what (formalised) mathematics prescribes. Instead, he argues that we will find a model for what is to be learned, in algebraic education, in what experts *actually do*; rather than presenting syntactical rules, we should allow students to experience and match algebraic manipulation: '...learning (is) always grounded in perception and pattern matching as embedded in practices' (page 95). However, rationality of the sort found in syntactical rules, he argues, is part of a different, social, process, a process to be understood as social legitimisation, and that we should not expect it to come '...from engagement with inherently logical artefacts.' Again, the role of the teacher is emphasised, though not directly. It is clearly a chapter dealing with epistemology and cognition, and the epistemology he proposes is one based on connectionism rather than on '(a) dualist philosophy (which) is the foundation of our culture's common sense about mentality' (page 88). This is an insightful chapter, which tackles old practices without simply discarding them as 'wrong,' choosing instead to discuss what is wrong with the environment in which they happen.

Chapter 6 'Transformation and Anticipation as Key Processes in Algebraic Problem Solving,' by Paolo Boero, extends in a certain sense and in a certain direction, the work of Arzarello, Bazzini and Chiappini (Chapter 4); the perspective is clearly epistemological. Overall, he proposes a way of reading what people are saying/doing when they engage in algebraic problem solving, a reading based on the notions of *sem* and *form*, while abandoning the traditional syntax/semantics distinction. Those two new notions allow one to apply the notion of *sense* (Chapter 4) both within (as in an example about the sum of four consecutive odd numbers) and outside the domain of mathematical meanings, in a powerful way; in many aspects it is also close to Chapter 3, by Lins. This is a very focused chapter which, through carefully selected examples, allows the reader to build an understanding of the implications of what the author proposes.

Chapter 7, by Aurora Gallardo ('Historical-epistemological Analysis in Mathematics Education'), takes on history as a reference, but again — as in Chapter 2 — not as a ready-made path for teaching. What Gallardo proposes is that history may provide a useful and important counterpoint to research in the classroom; the notions of cycles and interplay are crucial. Embracing Piagetian perspectives on cognition and learning, she sets out to investigate the similarities between what happens in the classroom and in history, acknowledging that 'The body of mathematical knowledge (...) is something that cannot be fully apprehended through its formal dimension...' (page 137). There are considerations both on how carefully chosen problems can further the process of acquisition of an algebraic competence and on student's responses to those problems. Insightful both from a historical point of view and for those looking for insights into ideas for development work, this chapter manages to combine a mainly didactical perspective with historical and epistemological ones.

Kaye Stacey and Mollie MacGregor wrote 'Curriculum Reform and Approaches to Algebra,' (Chapter 8) which in many ways seems to depart from the previous ones. Their particular concern has to do with discussing an assumption which has widespread acceptance in many countries: that generality — as emerging from generalising patterns — beyond being a root of algebraic thinking, also makes, in itself, a better route to it than other approaches. The implications are many, but crucially the authors draw attention to the need for criteria for making curricular choices, a point of particular interest for those involved in or analysing situations like the so-called 'maths war' in the state of California (USA), but also a point of permanent and general interest; in particular, they approach the relationship between research and curriculum reform. They conclude that '... the greatest use of findings such as those reported in this chapter (is) not to advocate one teaching method over another but to highlight the ways in which students think about mathematical situations' (page 152). If at first this chapter may seem a routine 'method-testing' exercise, it ends up at the other end: the students; starting from what seems to be a concern with teaching methods, they end up considering that '...curriculum designers are often concerned with how students ought to think instead of how they really think.' This is a challenging chapter, as it proposes that concerns with good teaching methods should be accompanied by making hidden assumptions and intentions explicit.

We then come to 'Theoretical Theses on the Resolution of Arithmetic-Algebraic Problems,' Chapter 9, by Eugenio Filloy, Teresa Rojano and Guillermo Rubio. Something outstandingly clear in this chapter is its *intention*: the carefully chosen and examined solution methods for algebraic problems place this chapter clearly on the didactical side. And there it stands, but it is the willingness to make students overcome a 'didactical cut' that guides the whole expedition; in particular, the didactical cut refers here to manipulating the unknown, as in the authors' previous work. Two classical solution methods — the arithmetic and the Cartesian methods — are described and examined, as well as two non-conventional methods. The main point is not about basic manipulative skills in algebra, but about *modelling*, although the focus remains throughout on examining how students deal with the relationships between the different pieces of data given in a problem. Epistemological and linguistic considerations are within the horizon of this chapter, which also draws

considerably from empirical work with students.

In Chapter 10, 'Beyond Unknowns and Variables: Parameters and Dummy Variables in High-School Algebra,' Hava Bloedy-Vinner looks at the use of literal notation in algebra, considering that the traditional concern with unknowns and variables has to be extended to an analysis of the role of parameters and dummy variables. Although students' work is used to illustrate the points being made, the line of inquiry falls clearly within the domain of mathematical meanings, that is, it is mostly concerned with what parameters and dummy variables are within mathematics itself, and with how students might interpret these new objects. She offers a number of questions which can serve to expose students' understanding of the notions of parameter and dummy variable. Her didactic recommendations are to use these questions in classroom discussions, and also to take care to discuss the role of each letter in expressions when manipulative activity or problem solving is taking place.

Giuliana Dettori, Rossela Garuti and Enrica Lemut wrote Chapter 11, 'From Arithmetic to Algebraic Thinking by Using a Spreadsheet.' In it they characterise algebra in intermediate school, and point out what determines the conceptual break from arithmetic, and analyse whether these features can be introduced by using a spreadsheet, focusing in particular on the elements to be taught and learned. In their approach they '...shift the focus from the potentialities of the software to the main characteristics of the subject to be taught, as (they) are convinced that, from an educational point of view, what matters is not to learn to find a numerical solution to algebraic problems but rather to understand the nature and the power of the theoretical solving scheme of algebra' (page 191). Through the use of a carefully crafted set of problems, the authors examine the advantages and disadvantages of using a spreadsheet for their solution, particularly with respect to the possibility of expressing and manipulating relationships. They conclude that there are evident limitations if one is aiming at introducing algebra, but also that ...using a spreadsheet (...) under the attentive guidance of a teacher...'(207) students can develop a number of important understandings related to the learning of algebra. This is a didactical chapter, placing great emphasis on the characteristics of the problems and also on teacher intervention.

Chapter 12 was written by Sonia Ursini: 'General Methods: a way of entering the world of algebra.' As the title suggests, this chapter shares a concern with the previous one, that is methods in algebra, through working with the computer environment Logo. Ursini's analysis, however, draws also from an examination of the history of mathematics, in particular the notion of *general number* as it appears in Vieta. Linked to this, Ursini proposes that the key difference between arithmetic and algebra is that, 'while arithmetic deals with numbers and an important aspect of it is to perform computations obtaining numerical results, algebra deals with general magnitudes and arises historically from the interest in deducing general methods for solving sets of similar problems'. Her work with Logo is centred on the procedural characteristic of the software, not the geometric one — as in other studies. Students use Logo to produce procedures to calculate, for instance, half of any number, and then Ursini proceeds to analyse what she calls 'the evolution of pupils' procedures,' to conclude that we may have here a powerful way of introducing students to the notion of 'general numbers,' one that she points out as essential for the development of algebraic thinking.

Chapter 13, the last one, is 'Reflections on the role of the computer in the development of Algebraic Thinking,' by Lulu Healy, Stefano Pozzi and Rosamund Sutherland. In it they discuss school algebra in the UK, the influence of Piaget on algebra research, Vygotsky and an 'algebrising' culture and the role of the computer. A central point in this chapter is a move towards a more directive approach to teaching, after the realisation that students would not 'naturally' engage in algebraic activity, no matter how suggestively algebraic the proposed task seemed to be. The authors associate this move with a change in theoretical paradigm — from Piaget to Vygotsky, put in a very broad way. The chapter ends with a statement of a current approach to algebraic education. The notions of 'setting' and 'tool' have become critically important, and an emphasis on the communicative function of symbol systems has emerged, leading to an approach to the learning and teaching of algebra which departs both from traditional symbol-manipulation and from an under-emphasis on the role of algebraic language in the development of algebraic thinking.

ACROSS THE BOOK

Many of the difficulties we encountered when working as a group related to views about what is and what is not algebra, and what is and what is not algebraic thinking. Whereas we may have disagreed on what algebra is, we did agree that it is important to analyse the different approaches which pupils bring to solving particular problems.

A related issue which soon emerged from the work of the group is that school algebra differs in its emphasis from country to country. However, to understand why this is the case would entail an investigation of the socio cultural influences in each country, which is beyond the scope of this book. What we wish to draw attention to here is that in some countries students are likely to engage with carefully crafted word problems which have traditionally been designed to engage students in constructing algebraic methods (see for example Chapters 9 and 11), while in other countries these types of problem have almost disappeared from the curriculum. As we have pointed out before, the meanings which students construct for algebra will be related to the types of problem which are prioritised in the mathematics classroom, and we think that the cross-national picture offered in this book might offer a contribution in this direction. Also, it is interesting to compare this approach with the earlier research on student's understanding of algebra (for example Booth 1984, Küchemann 1981), which first drew our attention to the multiple idiosyncratic meanings which students construct when they solve 'algebra problems', often explaining those meanings as being related to individual psychological development which would evolve almost spontaneously, whatever problems the students were engaging with.

Sometimes starting from the point mentioned in the previous paragraph, but not always, many of the chapters in this book are influenced by a socio cultural perspective to learning. This is particularly the case in Luis Radford's chapter (Chapter 1) which presents a detailed analysis of the historical origins of algebraic thinking, emphasising that the development of mathematics in ancient times was linked to the social, political and economic development of cities. This motivated a search for solutions to problems, such as how to calculate the area of a piece of land, how to solve inheritance problems and how to calculate the price of different commodities. In a similar way Aurora Gallardo (Chapter 7) discusses how the Chinese (ca. A.D. 250) were motivated to accept negative numbers well before other cultures, because of a need for these objects within problems involving 'gains' and 'losses'. Gallardo's iterative approach to analysing classical texts, linking the analysis to empirical work carried out with students, illuminates students' developing conceptions, while Radford, on the other hand, stresses that the social and cultural aspects in the development of algebra cannot be reproduced in the classroom. In any case, a historical analysis highlights what the axiomatic presentation of mathematicians leaves out, which is a discussion of both the practical and theoretical motives that lead to the resolution of certain problems, and the obstacles that are inseparable counterparts of the development of fundamental ideas (Gallardo, Chapter 7).

A socio cultural approach aims to understand how mental action is situated in cultural, historical and institutional settings (Wertsch, 1991, p 15). Whereas cultural differences between Western society and ancient or developing societies have often been interpreted as different stages in a normative development, both Radford and Gallardo show through their analysis that there is not a simple temporal progression in the development of mathematical language and thinking. What is crucial here is the dynamic interaction between the problems of the time and the technological, semiotic and epistemic tools available. This, however, is no simple relationship as the problems to be solved also drove the need for new technological and semiotic tools.

In his chapter Radford makes conjectures about the meanings which the ancients constructed from interactions with a particular combination of problems, tools and language. From a present-day vantage point it is very difficult for us not to view the notion of 'false position' from the perspective of the algebraic idea of the 'unknown' because the idea of an algebraic 'unknown' is now an established concept (together with a means of symbolisation). It is just this sort of difficulty which occurs when we analyse pupils' meaning construction as they solve a range of problems. It is difficult for the researcher not to ascribe meanings to pupils which are influenced by the researcher's own knowledge perspective. Whether we are analysing students' solutions to a range of problems or the solutions of an ancient historical group we will always be constrained by our own knowledge and perspective. This is why Lins (Chapter 3) stresses that any theoretical model has to be careful not to depend completely on what 'we' are in order to say what 'other people' are, or should be.

Still in the same direction, some authors advocate the introduction of computer-based symbolic languages as a type of intermediate language (tool) which can provide an entry point to the introduction of the algebra language. There is enough evidence that pupils who might find the algebra language alienating can more readily learn computer-based algebra language. However the new symbolic tool subtly changes the meanings which pupils construct and thus the associated learning. Moreover, computer-based approaches are often associated with mathematical modelling and problem solving and a focus on problem solving can work against the learning of algebra, because the focus becomes that of 'solving the problem' and not on the method for solving the problem, an awareness well developed within the group (for further discussion of this see the Royal Society/Joint Mathematical Council of the UK Report (1997) *Teaching and Learning Algebra, Pre-19*).

Mathematics learning centres around solving problems, and historical analysis tells us that the nature of the problems makes a difference to what is learned. But this is not enough because a particular problem can be solved in many different ways, as illustrated by the work of Filloy, Rojano and Rubio (Chapter 9), Dettori, Garuti and Lemut (Chapter 11) and Lins (Chapter 3). Lins suggests that pupils have to become aware of the different approaches and the epistemological limits of each approach.

Lins also stresses that the teacher should get pupils to make explicit justifications as a constitutive part of knowledge in order to allow shifts in meaning, that is construction of new knowledge. By establishing meaning through explicit statements it is then possible to provoke pupils to explore other meanings. Getting pupils to produce new meanings is what our school culture wants, and the role of the teacher comes in here to enable pupils to engage in producing meaning in a new way.

The traditional approach of presenting students with so-called algebra problems may have worked for some students when the teacher more or less imposed an algebraic solving approach. The current trend however is to encourage students to solve problems for themselves, without imposing a problem solving approach. We now know that students are likely to solve these problems in many non algebraic ways

which can result in an unresolvable tension. We have to ask the question 'can algebraic approaches develop in a seamless way from these non-algebraic approaches'. Some people believe that this is possible and others would suggest that the non algebraic approaches might even provide an obstacle to algebraic approaches. This is clearly an area where we need more research as curriculum reformers continue to fall into the trap of believing that 'good' problems alone will provoke algebra learning (as discussed by Stacey and MacGregor). The chapters in this book come together to show that 'mathematics problems' are only part of what would constitute an appropriate school algebra culture for students to learn algebra.

The complexity for the teacher is that pupils do already know arithmetic and so are bound to be influenced by these arithmetic approaches. The teacher has to know and understand these approaches if he/she is to make sense of student's constructions. The teacher has to understand the past history of learning which students bring to the classroom, which can emanate from both school and out of school experiences. As Wheeler (1996) has pointed out we have to take account of what students know but this does not then imply that we have to use an evolutionary model of learning. Instead he suggests that 'learning appears to me to be very largely a discontinuous, non-linear, business' (p 149). Reflecting on this issue would provide an appropriate backdrop to reading the chapters in this book.

NOTES

1 PME is the International Group for the Psychology of Mathematics Education.

distinguish some of the different conceptualisations between ancient numerical operations and the modern ones. Keeping this in mind, some of the steps of the translating solution include the following calculations:

$1/7[(4-1)x + (x+y)]10 = 4 \cdot (x+y), 3x \cdot 10 + (x+y) \cdot 10 = 28 \cdot (x+y), 3x \cdot 10 = 18 \cdot (x+y), x \cdot 10 = 6 \cdot (x+y)$

Then, the scribe chooses x=6 and 10=x+y and he arrives at y=4. (For a complete translation see Høyrup, 1993b). One of the points to be stressed here is the fact that the calculations showed in the previous sequence are based on an (implicit) analytical procedure: the scribe's calculations comprise the unknown quantities x, y (as seen in their own mathematical conceptualisation); the unknown quantities are considered and handled as known numbers, even though their numerical values are not discovered until the end of the process.

11 'J'ai mangé les deux tiers du tiers de ma provende: le reste est 7. Qu'était la (quantité) originaire de ma provende?' (Thureau-Dangin, 1938b, p. 209).

12 The problem of the transmission of algebraic knowledge and the sources of Greek (numerical and geometrical) algebra has been studied by J. Høyrup in terms of sub-scientific mathematical traditions. (see, e.g., Høyrup, 1990a).

13 The problem of whether a conceptual organisation is scientific or not is evidently a cultural decision. In the case of the Alexandrian algebra of the 3rd century B. C., it is hardly possible to ascribe to Diophantus the whole merit of building such a theory (Klein, 1968, p. 147). Nevertheless, we can say that, in all likelihood, his contribution was conclusive to this enterprise.

14 Freeman, 1956, fragment 4, p. 74.

15 When reading this quotation we have to keep in mind that Heath's translation is tainted by a modern outlook. Diophantus never spoke about 'negative terms'. Diophantus spoke rather of leipsis, i.e. of deficiencies in the sense of missing objects; this is why we might remember that a leipsis does not have an existence per se but was always related to another bigger term of which it is the missing part.

16 For a complete translation of the problem, see Heath, 1910, p. 132 or Ver Eecke, 1926, pp. 12-13.

17 This problem can be solved by the method of two false positions. Given that this method was invented later, we will not discuss it here.

18 Høyrup's translation of the problem-solving procedure is the following: '1 the projection you put down. The half of 1 you break, 1/2 and 1/2 you make span [a rectangle, here a square], 1/4 to 3/4 you append: 1, makes 1 equilateral. 1/2 which you made span you tear out inside 1: 1/2 the square line.' (Høyrup, 1986, p. 450).

19 Although the sign could be written in a stylised format, only a few variants were allowed. See Green, 1981, p. 357.

20 There were also female scribes, although, in all likelihood, they were not a legion! Such a scribe is the princess Ninshatapada (see Hallo, 1991).

21 For an example, see the solution to the problem No. 1, tablet BM 13901, note 18.

22 It is important to note that although Sumerian language was a dead language in the Old Babylonian period, in mathematical texts the scribes kept using some Sumerian logograms and, in repeated instances, they added phonetic Akkadian complements to some logograms as well. This rhetorical twist indeed shows a deep mastering of a very elaborated writing.

23 Note, however, that it does not mean that the scribes did the calculations by rote. Certainly, an understanding of what they wrote was part of the task of learning (some tablets show, for instance, that a good scribe was supposed to understand what s/he wrote).

24 It would be teleologically erroneous to think that the non-alphabetical cuneiform language of the Old Babylonian period was a delaying factor to the emergence of algebraic symbols in the Ancient Near East. The cuneiform language was a marvelous tool to crystallise the experiences, the meanings and conceptualisations of the people that spoke Sumerian and later Akkadian. Alphabetic languages correspond to new ways to see, describe and construct the word. One language is not stricto sensu better than the other: they are just different (For a critique of the alphabetical ethnocentric point of view, see Larsen 1986, pp. 7-9).

25 In the light of this discussion, it is easy to realise that it is an anachronism to see the development of algebra in terms of Nesselmann's three well-known stages: rhetorical, syncopated and symbolic (Nesselmann, 1842, pp. 301-306); further details in Radford, 1997

ROMULO CAMPOS LINS*

THE PRODUCTION OF MEANING FOR *ALGEBRA*: A PERSPECTIVE BASED ON A THEORETICAL MODEL OF SEMANTIC FIELDS

INTRODUCTION

Various characterisations of algebra and of algebraic thinking have been offered by different authors (for example, Arzarello et al., in this volume; Biggs & Collis, 1982; Boero, in this volume; Lins, 1992; Mason et al., 1985). Also, many articles, books and research papers have dealt indirectly with this issue. Choices made about what algebra and algebraic thinking are have a strong impact on the development of classroom approaches and material (Lins & Gimenez, 1997), that is, the discussion of this more theoretical issue is directly related to mathematics education in the classroom.

Each author makes epistemological assumptions — implicitly or explicitly, the former being much more frequently the case. Almost all those sets of assumptions have at least one common feature, inherited from traditional epistemologies. The first part of this chapter deals with the analysis of that common feature, arguing that it does not allow a sufficiently fine understanding of the process of production of meaning for algebra. On the basis of this analysis, a new characterisation of knowledge is produced, leading to an epistemological model in relation to which algebra and the production of meaning for algebra are, then, characterised.

The second part of this chapter consists of the examination, from the point of view of the theoretical framework developed in the first part, of two situations in which the production of meaning for algebra occurs — actually or fictionally. The discussion proposed in this chapter is about ways of conceptualising cognitive activity, and the role of the little empirical evidence introduced is simply to provide a vehicle for this discussion.

For the purpose of keeping a sharp focus, I will always use examples related to quite simple linear equations; I hope the reader can see in them 'exemplary examples.' The design model presented on page 51 is subject to the criticism that it

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is artificial; I would certainly agree with this, and with the suggestion that realistic or real-life situations should be part of mathematical education in school. Nevertheless, we should not forget that many times what starts as artificial becomes quite real for pupils, even more so with the younger ones; also, many aspects of mathematical education in school are not 'naturally' found 'in the streets,' and I think algebraic thinking is one of these.

RE-THINKING EPISTEMOLOGY

On familiar ground

Saying what algebra 'is' is not a minor problem, nor one without significance within mathematics education. But we can avoid this discussion for a while, and start instead examining a much less controversial situation.

I am quite sure that everyone in the mathematics education community will agree that solving an equation such as '$3x+10=100$' is algebra. If for no other reason than because one has to deal with an 'unknown' expressed in literal notation, and dealing with literal expressions of this kind is algebra, even if it is not the whole of the subject.

How can the task of solving '$3x+10=100$' be fulfilled? Certainly in a number of different ways:

(1) Try different numbers, until you (hopefully) get it right.

(2) Think of 3 boxes which, together with a 10kg weight, balance a 100kg weight.

(3) Think of a number, multiply it by 3, and add 10; the result is 100. Now undo it.

(4) 3 parts of a value as yet unknown, together with a part of value 10, compose a whole of value 100.

(5) Add or subtract the same number from both sides; multiply or divide by the same number. Aim at an expression of the form $x=\ldots$.

All these approaches are, in fact, so familiar that many of us tend to take them as being only different appearances of the same essence, with the likely exception of (1). I will however argue that this is not the case.

We start by characterising what the equation 'is' in each case.
In (1) it is a condition to be fulfilled.

In (2) it is a scale-balance.
In (3) it is the do-list of a function machine.
In (4) it is a whole-part relationship.
In (5) it is a relationship involving numbers (including x), arithmetical operations and equality.

What can be done with a condition to be fulfilled? Nothing, apart from substituting numbers and checking. If the condition is changed everything changes.

What can be done with a scale-balance preserving the equilibrium? Well, a lot, for instance adding the same to both sides; or removing the same from both sides, given there is enough to be removed. Secondly, and as a consequence, one can double what is on each side, but this is not necessarily as visible as the two other operations. The equilibrium is also preserved if we have 2.7 of what was on each side, but that is even less immediately visible.

And with a function machine? Undo. One can, of course, use it as a condition to be fulfilled. What does not make much sense — if it makes any at all — is to have a result, the number on the right side, expressed in terms of the number one is seeking; this would be the case with the equation '$4x+10=x+100$.' (Fig. 1) Strictly speaking, although '$3x+10=100$' is a natural for function machine, '$100=3x+10$' is not; even more disturbingly, '$10+3x=100$' is not natural either.

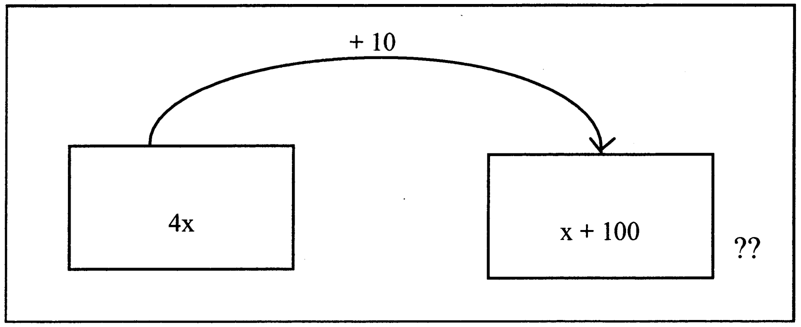


Figure 1

With respect to whole-part relationships, many things could be done: separate the parts (decompose the whole), for instance. Or compare that whole with some other; or make a part into a whole.

Finally, as to the object in (5), one can check any textbook on school algebra.

For someone who is acquainted with these five possibilities for producing meaning for '$3x+10=100$,' it is possible to speak of metaphors and of switching from one to another. But we can, instead, think of a child who is presented for the first time with '$3x+10=100$,' and told 'it is a scale-balance.' It is likely that the child will learn to deal with that situation, and to use the 'add-remove-share' approach to solve similar equations. The question now is this: what would s/he say of '$3x+100=10$'? My guess is: 'that one can't be,' as a student wrote in a similar situation, in a study we conducted (Lins, 1992), and the reason is quite simple: there cannot be a balanced scale-balance with 3 things and 100kg on one side and only 10kg on the other.

If, instead, the child is told that 'it is a function machine,' and supposing that this is a notion already familiar for the child, a completely different situation arises. Now it is perfectly possible to produce meaning for '$3x+100=10$,' but not for '$4x+10=x+100$,' particularly within an activity aimed at solving this equation.

Three points arise. First, the text '$3x+10=100$' can be constituted into objects in at least five different ways[1]. Second, depending on the objects constituted, there will be a certain logic of the operations, that is, peculiar ways of handling those objects, things which can be done with them. Third, and crucially important for mathematical education, there are other equations for which it is not always possible to produce meaning in ways similar to those possible for '$3x+10=100$'. The impossibility of producing meaning for a given statement is what I call an *epistemological limit*. This is a useful notion, as it points to the fact that producing meaning in relation to, for instance, a scale-balance, is not always a metaphorical act. A remarkable instance is the impossibility in Greek mathematics of producing meaning for incommensurability as related to numbers. For some authors the separation between geometry and arithmetic is, in Greek classical mathematics, simply a trick to avoid technical problems; however, Jacob Klein (Klein, 1968) has conclusively shown that this is not the case, and that the separation is fully consistent with the ways in which meaning for number and for geometric objects was produced in Greek classical mathematics. (see also: Lins, 1992)

Knowledge

To approach the problem of knowledge, we consider two people who have produced meaning for '$3x+10=100$,' one of them in relation to a scale-balance, the other in relation to *algebraic thinking*[2]. Both would plausibly state that 'one can take the same (10) from both sides.' For the first subject, it would be so because 'if the same is removed from both sides of a balanced scale-balance the equilibrium remains,' while for the other it would be so because 'one can add any number (-10, for instance) to both sides of an equality, and preserve it.' The question here is not, of course, whether or not both of them will do the same thing, but why will they do so in each case. The key problem becomes, then, is it adequate to say that both subjects

share a knowledge? To make things more dramatic, we might want to consider a five year-old child who says that '2+3=5 because if I put two fingers together with three fingers I get five fingers,' and a mathematician who also says that '2+3=5,' but with a justification built from Set Theory. I think in both cases the answer is *no*, it is not adequate, but there is a further difficulty with that inquiry: we did not say what we mean by 'knowledge.' Such a key question has been many times overlooked, possibly because there is a strong traditional view about the subject, or possibly because people do not see this as a relevant question within mathematical education, but most likely a combination of the two.

Although many reformulations of the original notion have been produced, traditional understanding of knowledge is still bound to a classical definition:

> *A person is said to know that p if: (i) that person believes in p; (ii) p is a true proposition; and, (iii) the person has arrived at proposition p through an acceptable method, that is, the belief is justified. (See, for instance, Dancy (1993, ch. 2))*

The above definition is usually taken as saying that to know is to have a justified true belief. Let's examine some consequences of this definition.

First, there has to be some knowledge-independent criterion of truth; some alternatives offered in this direction are objectivism, Platonism and Cartesianism. Second, and this is a very interesting aspect of the classical definition, saying that a method is acceptable means that someone judges it acceptable; for some people casting shells is not an acceptable method for forecasting the weather, but for others it might well be. The implication is that 'knowing' is a socially constructed situation, although 'knowledge' — according to it — is something of an absolute nature. With respect to the classical definition, justifications have to do with the right of a person to say s/he knows, but not with the constitution of 'knowledge.'

The classical definition is troublesome, as shown by what is called the Gettier Problem, a construction in which someone would be granted — rightly, from the technical point of view — the knowing of something s/he does not know; the argument leads to the fact that the three conditions for knowing are not sufficient (Gettier, 1963). There is also the criticism, of a non-technical nature, that the classical definition rules out *implicit* knowledge; I will return to this later.

From the classical definition I want to emphasise the fact that knowledge, according to this definition, has the status of a proposition, being 'that which one knows' (for a very good and accessible discussion on the traditional view of 'knowledge,' see, for instance, Ayer 1986, Chapter 1); in fact, this is true also for the practitioners of the 'implicit knowledge' idea.

Saying what knowledge *is* is not, of course, a matter of finding out the truth, but rather a matter of conceptualising things in a way which produces useful insights which to some extent agree with our general experience on the subject. We must then ask: is 'knowledge is that which one knows to be the case' a good definition? According to that definition, we must say that the child and the mathematician share a knowledge, namely that '2+3=5.' A critical instance emerges if we consider the case of a very young child who confidently says that '2+3=5,' because his father — a fine mathematician — told him. The child believes what he is saying is true, it is actually true, and his father is a credited authority in the field; the fact is, however, that the child is not talking about numbers — not even 'finger numbers,' but we would be led to say that the young child shares a knowledge with his father, the mathematician. Clearly, 'knowledge is that which one knows' is not an adequate notion.

That the notion of 'implicit knowledge' relies on such an inadequate notion, can be shown by observing that someone has to *say* that a person has this or that 'implicit knowledge,' that is, at some point it assumes a propositional form, and because the person one is talking about is not aware of having the said knowledge, all we have is that proposition. If that requirement is dropped, that is, if we do not require that someone *say* that a person has this or that 'implicit knowledge,' the difficulties are even greater, as we should come to the conclusion that we all know all the things as yet unimagined by any human being. Also, we must be aware that ordinary language may be very flexible with respect to the uses of the verb 'to know.' Not all we know is knowledge; for instance, I might say that 'I know John,' but that does not mean 'John' is knowledge. In a similar fashion, to say that someone knows-how (to do or to make) something is different from saying that a person knows-that.

We can now highlight what seems to be three key aspects of knowledge.

First, the person must believe in something if that is to constitute part of a knowledge s/he produces, and that implies s/he is aware of holding that belief.

Second, the only way we can be sure of that awareness is if the person states it, and here I am using the term 'state' freely, meaning some form of communication accepted by an interlocutor; it does not have to be linguistic in form.

Third, it is not sufficient to consider what the person believes and states, as different justifications with the same statement-belief correspond to different knowledge. Moreover, justifications are related to what can be done with the objects a knowledge has to do with; in the case of the child saying that '2+3=5,' for instance, '2+3' is the same as '3+2,' once the arrangement of the fingers does not make any difference. From the point of view of a set-theory based justification, spatial arrangements are not something having to do with '2' and '3' or with their addition.

Justifications, then, play a double role in relation to knowledge. First, they are indeed related to the granting of the right to know, and this granting is always done by an interlocutor towards whom that knowledge is being enunciated. Second, they are related to the constitution of objects.

Within the view I propose, *knowledge* is understood as a pair, constituted by the stated proposition which one believes to be true (the *statement-belief*), together with a *justification* the subject has for holding that belief:

knowledge = (statement-belief, justification)

A justification has to fulfil the double role indicated above: it has to be acceptable (for some interlocutor), at the same time as it constitutes, for the subject of knowledge, objects, that is, s/he can say something about those 'things.' Moreover, the definition establishes that there is always a subject of knowledge, rather than simply a subject of knowing, as in the traditional views; it also establishes that knowledge is produced as it is enunciated.

I am not saying, of course, that knowledge is all there is to human cognitive functioning; knowledge is part of it, a substantial and accessible part. I am saying, indeed, that knowledge is characteristically human, as sign-mediated activity is.

Algebra

As the main purpose of this chapter is to discuss the production of meaning for algebra, my next step will be to give a characterisation for algebra; given the previous discussion, it does not seem reasonable to say that algebra is knowledge. Let's see why.

We may start noticing that we would naturally say that '$3x+10=100 \Rightarrow 3x=90$' is algebra. But we have also seen that it is possible to produce meaning for that statement in a number of different ways; if that is to be knowledge, there is at least the justification missing.

Second, it is true that, generally speaking, we identify algebra with statements *potentially* interpretable in terms of relationships (equality, and eventually inequality) involving numbers and arithmetical operations; in order to say that '$3x+10=100 \Rightarrow 3x=90$' is algebra, we do not make reference to which meaning is being actually produced for it.

Instead of saying that algebra is knowledge, we would do better to say that it is a set of statements with the characteristics described in the preceding paragraph. Notice, however, that we know nothing about the meanings which will be produced for them by a given person, in a given situation; they may well be related to a scale-balance or to a function machine. That characterisation of algebra is operational for the purposes of research; in a later section I will show that it is also operational for the purposes of development.

Algebra being a set of statements, it is (for me) text (cf. Lins, 1996). Producing meaning for a statement of algebra is producing meaning for a text[3], and producing meaning for a text is to constitute objects from that text and relationships between them. Justifications, as an integral part of knowledge, play a role in the constitution of objects from the text of the statement-belief.

The characterisation of algebra I propose is, then, operational in two aspects. First, in pointing out that sharing statement-beliefs in algebra is not enough evidence of sharing knowledge. Second, in pointing out that we should examine justifications if we are to identify the objects being constituted from the statements of algebra.

Semantic Fields

The notions of algebra and of knowledge I have presented so far enabled us to clarify two things. First, to distinguish algebra from knowledge in which algebra contributes statement-beliefs; this is an important distinction because it allows us to account for different meanings produced for algebra. Second, it draws attention to the fact that 'people dealing with algebra' is a demarcation heavily marked by our own system of categories, and that we should take this into consideration when examining other people's activity of producing meaning for algebra.[4] But there are further consequences.

For instance, if Seeger's quite interesting question (Seeger, 1991, p. 138) 'How do teachers convert content into forms of interaction and how do students convert those forms into content,' is reasonably reframed to 'How do teachers convert knowledge into forms of interaction and how do students convert those forms into knowledge,' it becomes clear that it is the dual role of justifications which play the key role. On the one hand, justifications are interactional in nature, i.e., knowledge is always produced *through* interaction — be it physically or remotely established — and *aiming at* interaction (Lins, 1996); on the other hand, by establishing objects they produce 'content' by producing meaning. The question now becomes: 'Is there non-interactional learning?' I am not asking, of course, whether someone can learn alone in a room; I am asking whether such lonely learning is or is not actually interaction-free in a wider cognitive sense.

When new knowledge is produced, it can be new in two ways. It can be new in that the belief stated was not a belief before its constitution into knowledge. But it also can be new in that a new justification is produced for a statement-belief which had already been part of another knowledge, with another justification.

First we consider the case of someone who has produced meaning for '$3x+10=100$' as a balanced scale, and then enunciates that (K_1=) 'I can take 10 from both sides and preserve the equality, because it is a balanced scale.' S/he now enunciates, for some reason (related to the person's present activity), that (K_2=) 'I can add 90 to both sides and preserve the equality, because it is a balanced scale-balance'; that might be new knowledge, in case the person did not hold, before its enunciation, that stated belief in relation to the object he has constituted from '$3x+10=100$.' Notice that the two justifications are produced in relation to the same kernel, involving a scale-balance.

Alternatively, we consider that after producing K_1 s/he might enunciate that (K_3=) 'I can take 10 from both sides and preserve the equality, because it is like two equal piles of stones.' K_1 and K_3 have the same statement-belief, but justifications which are not only different, being in fact produced in relation to different kernels (involving a scale-balance in the first case and piles of stones in the second).

In order to provide a more flexible and complete account of such differences in knowledge production, we need another construct, which I call a *Semantic Field.* We will say that a person is *operating within a given Semantic Field* whenever s/he is producing knowledge (meaning) in relation to a given kernel; we will refer, for instance, to someone operating within a Semantic Field of wholes and parts. Alternatively, we may say that a Semantic Field is the activity of producing knowledge in relation to a given kernel. A kernel may involve a scale-balance or piles of stones, but also wholes and parts, function machines, a straight line, areas, money, a thermometer, algebraic thinking, all sorts of fantastic creatures, colours; indeed, it may be composed by anything conceivably existing. What is known about the kernel is not 'justifiable' within that Semantic Field; they are *local stipulations.* I am just extending Nelson Goodman's notion of a stipulation (Goodman, 1984; Bruner, 1986). Although *local stipulations* are 'given' within a certain meaning producing activity, they are not necessarily basic in the strong sense proposed by Goodman for his stipulations, that is, they might well be questioned, challenged or even provided with *justifications* within some other *Semantic Field.* We could perhaps say that reality is a *Semantic Field* with a *kernel* constituted by stipulations in Goodman's sense.

The notion of Semantic Field allows us a dynamic view about meaning production. On the one hand, we are able to consider how — and if — new knowledge comes or not to be part of a transformed kernel. On the other hand, we are able to consider how — and if — knowledge is produced in relation to a given kernel relate to each other. Moreover, and this is a key aspect, we are able to make full operational use of the notion of epistemological limit, already mentioned.

By an epistemological limit I mean the impossibility of producing meaning for a statement within a given Semantic Field; for instance, it is impossible to produce meaning for the text '$3x+100=10$' as a balanced scale-balance. The operational importance of this notion is to establish that: (i) every time meaning is produced there is a restriction on the horizon for further meaning production, implying that, (ii) if learning is understood — rightly, I think — as learning to produce meaning, teaching must also aim at an explicit discussion of the limits created in that process. In a later section I discuss the relevance of this construct — epistemological limit — to development and the classroom.

The first of the two possibilities considered a few paragraphs above (the one involving K_1 and K_2) I call a *vertical development*, the constitution of new statement-beliefs into knowledge within the same Semantic Field. The second possibility (the one involving K_1 and K_3) is a *horizontal development*, the production of knowledge from the same statement-belief, but within a different Semantic Field.[5]

A horizontal development between Semantic Fields S_1 and S_2 always implies a vertical development within S_2. Also, a horizontal development characterises either the establishment of a metaphor ('3*x*+10=100' (an arithmetical relationship) is as if it were a balanced scale') or of a reversed metaphor ('a balanced scale is as if it were an equation (an arithmetical relationship)').[6]

The constructs *knowledge* and *Semantic Field* form the core of the *Theoretical Model of Semantic Fields* (**TMSF**). On this basis we can speak of producing meaning for *algebra* (a *text*) as the production of knowledge from algebra within Semantic Fields. Operating within different Semantic Fields means constituting *objects* to which particular *logics of operation* apply. New knowledge can be produced through vertical and horizontal developments.

Interlocutors are the source of legitimacy for knowledge, and truth is relative, but not 'absolutely relative.' From the point of view of the **TMSF** truth is not a notion to be applied to the statement-belief, to the proposition which we know to be the case, but to knowledge, which implies that truth is a cognitive notion, and not objectively related to 'hard facts.' To be able to decide whether or not a statement is true, certainly one must make a decision on what is being talked about; but 'what is being talked about' is constituted precisely through knowledge enunciation, and truth is, thus, relative. Justifications have a role to play in the establishment of truth, and once justifications are always produced towards interlocutors, the 'individual' cannot any longer be taken as a source of truth, as assumed, for instance, by radical constructivism. What is produced is a relativism which has cultures, through the many practices which compose them, as the domains of relative validity of any given truths.

Within the **TMSF** the distinction between algebra and algebraic thinking becomes natural. Moreover, thinking algebraically is to be seen primarily as a consequence of cultural immersion.

GLANCING AT MEANING PRODUCTION FOR ALGEBRA

In this second part, I will present and discuss some empirical material and a design model for classroom activity which has been tried with pupils; all the material presented is intended only as a vehicle for discussing the notions in the **TMSF**, giving the reader a chance to see how the notions proposed 'work in practice.'

Simple word problems

As part of a wider research and development project[7], we have separately interviewed two pupils, presenting them with word problems. Our main objective was to investigate two things: (i) the objects with which pupils were operating; and, (ii) the role of interlocutors in the process of producing knowledge related to the solution processes.

The strategy adopted was to question pupils as to the justifications they had for making the statements they did; in the particular case of the problems we will discuss, the basic 'solution' statement always had to be a choice of operation which solved the problem proposed. Secondarily, there were other statements related to the justifications offered by them.

The two pupils interviewed studied in the same state school in Rio Claro, Brazil. FEE, a girl, 13years old, and FAB, a boy 12years 7months old. The interviews lasted about one hour, during which they solved three or four problems; only one of those problems is discussed here, the **Oranges&Boxes** problem:

(1) To calculate how many oranges will fit into each box, we divide the total number of oranges by the number of boxes, i.e.,

$$\textbf{oranges per box} = \frac{\textbf{number of oranges}}{\textbf{number of boxes}}$$

(a) If I tell you the total number of oranges, and the number of oranges in each box, how would you calculate the number of boxes used?

(b) If I tell you the number of oranges in each box, and the number of boxes, how would you calculate the total number of oranges?

The reason for presenting the 'algebraic' formula was to ascertain whether the pupils would constitute it into an object, dealing with it in the process of solving the problem; neither of them made any reference whatsoever to this formula.

At first FEE seems confused about what is given and what is not. After a somewhat long exchange, she says

> **FEE** Then...for example, there are 40 oranges, I divide byyy...(softly) Wait a minute...(reads the problem again, very softly)...Yeah...I divi...then I would divide the...the total num...you'll, for example, you...for example, I have 10 boxes, 4 go into each box, then I divide the total (number) of oranges by the total of...how many go into each box (writes down: 'I would divide the total of oranges, by'; looks to the interviewer) By the oranges, isn't it?

I What is it you want to say?

FEE That I would divide the total of oranges by the oranges that go into the box. (and writes down: 'how many oranges go into **each box**.') Can I move to b?

We asked her about justifications:

I ...why is it that calculation (a division) and not another?

FEE Because I am, for example, I am, I am, I have to divide among the boxes, got it, to know ('the number'; said together with the interviewer's next question)

(...)

FEE So, but you told me how many go into each box, OK...I would divide them...for example, you say that 4 go into, there are 40, I would divide them (the 'I would divide them' is accompanied by a gesture: open hand, palm down, touches the table as if indicating 'lots'), got it...?

I Divide means what? Sharing, you're doing?

FEE Yeah, for example I put 4 in a box, 4 in the other, 'till it's finished...then I would know how many...

(...)

I (...)...you're saying you thought this way, but you did a division...How...why did it occur to you to make a division?

FEE Well, because I had for me to, for example, for me to...to know how many...how many will go, for example, you have 3...have...20 oranges to put each...to put 5 in each box, then I'll have to divide, got it...? I can't multiply it will increase the oranges, got it? Nor add, nor subtract...

I Why not?

FEE Because I can't!! (laughs) How will I add? Look, there are 38 oranges, I will add to what? (One) has to divide with the boxes, I can't add.

A number of things emerge. FEE used specific numbers, but they were always 'number of something,' boxes, oranges, oranges per box, and both elements are relevant. It seems that the role played by the chosen numbers is to check a reasoning in which the logic of the operations is primarily related to sharing oranges into boxes; there is an interesting interplay between using 'divide' to refer to an arithmetical operation and to sharing. Moreover, the numerical division and the sharing are so closely bound that she has difficulties in explaining why she did a division when she was 'in fact' doing a sharing; the explanation she gives reveals that both realistic constraints ('I can't multiply, it will increase the oranges!') and

dimensional constraints imposed on the quantities involved were part of the kernel in relation to which meaning was being produced.

FAB, the other interviewee, was presented with the same problem. Prompted to 'think aloud,' he said,

FAB I thought...I didn't see this bit here...I want to know how many oranges go into, then I...how many oranges would take...and would divide...I would take all the oranges and divide...by the number of boxes...(frowns) No...no...I wouldn't have the quantity of boxes...

I OK, then...but you want to know the quantity of boxes...what do you know? You know how many oranges go into each box, how many oranges altogether, you want to know how many boxes you need...

FAB ...(looking at the problem) I would take the whole and put...I would divide by the number of bo...of oranges that go into...

FAB (reads what he had written down) 'I would take the quantity of oranges and divide by the quantity of oranges that go into each box.'

I Are you sure that you would get the number of boxes...

FAB (nods, and moves to (1b))

I What did you think that took you to this conclusion?

FAB (smiles) The...(smiles)...I...

For a while he did not say anything, so we asked him to play as if he had to explain to a cousin why he did the problem that way; FAB said it couldn't be because his cousin was too young to go to school. We decided to adopt 'a friend' instead of the cousin.

I Imagine your friend is there, at your side, and he asks 'Listen, FAB, how do you know it? How did you think to...solve the problem?'

FAB ...Well, I thought if I had the oranges with the box.

I Hmm. Try to show me...How did you imagine it? Try...if you want to make a drawing, anything with the hands, to speak, whatever you want.

FAB (drawing round shapes on the paper) I imagined I had a pile of oranges.

I Hmm.

FAB Then...I took a box (draws a square 'u')...which would hold...a certain amount (draws some round shapes inside the 'box') Then I thought 'If I divide this amount of oranges (points to the shapes outside the box) by the amount which is inside (points to the box)...which goes into here, I'll find out.'

FAB's solution is substantially different from FEE's in at least one key aspect: he *never* mentions any specific numbers. The objects he is operating with are related to boxes and oranges. More precisely, he seems to constitute the following objects: unit oranges, a pile of oranges, and a box. Those objects have properties; for instance, the pile of oranges can be counted or separated into boxes. The logic of the operations engendered by his construction did not depend on specific numbers. It seems he thought of the operation 'separate the oranges in the pile into smaller, equal, groups,' and only then indicated the arithmetical operation division which, as a tool, would be used in order to do an actual calculation. The use of an arithmetical operation is subordinated to the logic of the operations proper to the objects constituted. The boxes-oranges kernel is so solid that when first approaching the problem he says 'I would take the whole and **put**…,' and immediately changes to 'I would **divide by** (…)' (my emphasis); he is thinking around a kernel of boxes and oranges, but the problem says 'how would you *calculate*.'

Another aspect of interest is that, although he had already hinted that he was operating with 'boxes and oranges' objects, it took a long exchange before FAB felt he could give the justification he apparently had in mind, and it is remarkable that this exchange involved precisely a proposed change in interlocutor, from the interviewer to a peer. The slip-of-the-tongue 'I put,' provides a strong indication that the justifications given later were not some 'rational reconstruction,' but rather a true enunciation, to a new interlocutor, of a faithful account of the 'actual' solving process.

FAB also solved (1b) correctly, but this time it took him no time to produce a justification within a Semantic Field of oranges and boxes:

I The same thing: how would you explain it to your friend?

FAB I had…as I imagined, that I had the boxes, some ten boxes. Then I would do the opposite to that (points to (1a)). I would take as if I was going to count, but instead of counting I would find out how many oranges in each box and would do times. By the number of boxes used.

…

I Then you did a multiplication…Why?

FAB Because it's quicker, isn't it, than counting one by one.

I But it is the same thing you're doing, just that to calculate…

FAB Yeah.

A possibility for FAB's need to specify 'some ten boxes,' is that in the ordinary experience of most people with boxes there are never too many and these can be precisely quantified without difficulty, while with respect to piles of things one is almost never expected to have a precise or near-precise idea of the quantity; put in other words, *piles of oranges and sets of boxes may have had different properties with respect to quantification.* In solving (1a) it is just natural that the number of boxes is not determined at first, and the other objects (a pile of oranges and lots of oranges to be put into boxes) do not suggest the need of a perceptual at-a-glance quantification.

FAB clearly constituted a distinction between the operation actually carried out (counting of a number of equal lots) and the arithmetical operation used as a tool to evaluate the result of the counting; FEE, however, did not.

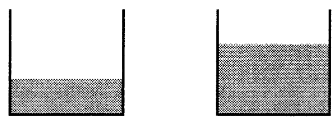
Altogether, the interviews showed us different processes of meaning production for a text involving oranges and boxes. Numbers were constituted as different objects in each case, that is, they had different properties and played different roles. Distinct logics of operations were in place, but in neither of the two cases properties of the arithmetical operations played a part in these logics of operations, that is, arithmetical operations were not made into objects.

A design for explicit justifications

In this section I will argue that justification, as a constitutive part of knowledge, has to become an explicit part of classroom environments. Presenting pupils with 'problems to solve' will focus the activity on producing a solution, and it is only natural that trying to get them to discuss their methods starting from a solution-driven problem requires some considerable effort on the part of the teacher.

Classroom common-sense, built both from tradition and from some scientific common-sense, suggests that the natural direction is from 'concrete' to 'abstract'; one possible interpretation here is that 'concrete' may refer, for instance, to problems with specific numbers, while 'abstract' would refer to problems with generic numbers. Freudenthal has already argued against such conception, pointing out that it is not true that generality is always achieved through generalisation (Freudenthal, 1974). More particularly, this comment was prompted by an analysis of Soviet work on the early introduction of generic, literal, expressions to pupils, and in the cases he examined, 'early' meant the first grades of primary school.

If we take in particular the pioneer work of V.V. Davydov (e.g., Davydov, 1962), the most striking feature is a conceptual shift through which he departs from the accepted notion that 'numerical literals' (our terms) can only be made meaningful as 'variables', as generalisations of specific numbers. What Davydov proposes is to work from generic quantitative relationships, as one would find in a situation involving cars and trucks in a parking lot: 'In a parking lot there are two kinds of vehicles: some are trucks and some are cars. If all the cars leave, which vehicles are left?'



If 9 buckets of water are poured into the tank on the left, it will become full.

If 5 buckets of water are poured into the tank on the right, it will become full.

Figure 2

As we have already indicated in Lins (1994), Davydov's work is based on the notion that even if the original 'support' is provided by a trucks and cars situation, what one is **in fact** dealing with is the 'true' nature of simple algebraic relationships, i.e., pupils are given the chance to work with an embodied version of the essential whole-part relationship. From the point of view of the **TMSF**, however, what Davydov is proposing is that pupils produce meaning for literal expressions within a Semantic Field of cars and trucks, and then use these expressions as a departure point for beginning the development of Algebraic Thinking, as meaning for new statements gradually comes to be produced in relation to statements already made meaningful, rather than in relation to the original situation (kernel). This shift is not treated explicitly in Davydov's activities, but it could have been.

Based on Davydov's original idea, I have developed an activity in which literal expressions are made meaningful within a given Semantic Field (of water tanks), and then a deliberate shift in the way meaning is produced for new expressions generated is proposed to pupils. This activity has been tried with sixth-graders in Brazil, and an overview of the results is presented in Lins (1994). I will not be discussing here actual pupils' work, but the overall shape of the approach proposed. The activity is introduced with the following text.

What is being proposed is that the tanks situation will constitute a kernel in relation to which meaning will be produced for various statements. Objects constituted in that kernel are — or could be — buckets and tanks. Implicitly, there is also water, or some other liquid; another local stipulation, suggested by the drawing, is that the two tanks are of equal size.

One possible first statement is,

S_1 'The tank on the right has more water than the tank of the left.'

There are, however, at least two different justifications for the enunciation of this statement-belief:

J_{1A} 'The line of water is higher on the tank on the right,' or,

J_{1B} '9 buckets are needed to fill up the tank on the left, but only 5 are needed on the right.'

Within the **TMSF**, the knowledge K_{1A} = (S_1, J_{1A}) is different from K_{1B} = (S_1, J_{1B}). The difference is not just a formal one; in K_{1A} the drawing itself becomes an object which has a place in cognition, while in K_{1B} there is only reference to the 'verbal' part of the text. Notice that we are not claiming that the subject enunciating K_{1B} has never or will never constitute the drawing into an object; it is just that in the enunciation of K_{1B} that object does not seem to exist. While K_{1A} seems to have a more qualitative nature, K_{1B} seems to have a more quantitative one.

To appreciate more fully the characteristic difference between K_{1A} and K_{1B}, we may consider whether similar justifications could be used in relation to the following question: 'If I add one bucket of water to the tank on the left, will there still be less water in it than on the tank on the right?'

Operating with a quantitative relationship there would be no difficulty in providing an affirmative answer.

S_2 'If one adds a bucket of water to the tank on the left, it will still have less water than the tank on the right.'

J_2 '8 buckets will still be needed on the left, but only 5 on the right'

Operating only with an object constituted from the drawing, however, it is impossible to produce an answer. It is possible, of course, to consider that the top 'white' space on the right corresponds to 5 buckets, and to estimate the 'slice' corresponding to a bucket, using that estimate to conclude visually that there would still be more water on the right than on the left. That operation, however, depends on also constituting 'a number of buckets' as an object.

Returning to the activity, I would propose that the pupils produced valid statements together with justifications. It became more comfortable to assign single-letter names to the objects being referred to; thus, 'buckets' became 'b,' and equality was naturally indicated by '='. As to the tanks, we finally agreed on 'T'. The amount of water in the tank on the left was named 'X', and that in the tank on the right, named 'Y'. All these choices were made together with the pupils, and they seemed to have added no further difficulty to the activity. The pupils in question were Brazilian sixth-graders (12-13 years old), who would have had by then an introduction to simple equations using x's or y's.

One might expect to get expressions like 'X+9b=Y+5b,' which are **not** to be understood as 'equations.' At first it is natural to get justifications which all refer back to the kernel, for instance:

S_3 'X+9b=Y+5b'

J_3 'If 9 buckets are added to X, we will get a full tank, and the same happens if we add 5 buckets to Y.'

Or,

S_4 'X+4b=Y'

J_{4A} 'If 4 buckets are added to X, there will be 5 buckets missing on the left, and this is what is missing on the right.'

The enunciation of the knowledge (S_4,J_{4A}) produces meaning for S_4 within a Semantic Field of tanks. Meaning for a statement is produced by the constitution of objects from that text.

One could also produce the following justification for S_4:

J_{4B} 'If 9 buckets are missing on the left, and 5 buckets missing on the right, this means that Y has 4 buckets more than X.'

which is, of course, different from J_{4A}.

The reason for eliciting justifications which refer back to the kernel is that the statements must first be made to correspond to objects already constituted, and the kernel, with its objects constituted through local stipulations plays the psychological role of reality; that is, of course, an approach radically different from objectivist theories of meaning, for which 'hard core reality' objects are the things in which meaning is 'anchored'.[8]

Once meaning is established for a set of statements, within a Semantic Field of tanks, it is possible to suggest that pupils explore another way of producing correct statements about the tanks situation, and this is done by examining possible relationships between already established statements: how could one 'reach' the statement 'X+4b=Y' — already meaningful — starting from the — already meaningful — statement 'X+9b=Y+5b'? Our algebra-educated minds would certainly say, quite naturally, 'Take 5b from each side.'

Before that can be taken as a natural step, however, we must consider that the very task proposed involves two crucial steps: (i) that the statements themselves become objects; and, (ii) that pupils' thinking shifts quite strongly from the tanks situation. Let's examine the consequences of that.

First, in order to constitute the statements into objects, we must be able to say something about the properties they have *as whole statements*; it is not enough to say what their constitutive elements *are* nor what the statement says about those objects. But pupils are precisely being required to say something about what does not exist yet for them, *statements as objects*. There seems to be an epistemological paradox here.

Second, any direct transformation from 'X+9b=Y+5b' to 'X+4b=Y' will produce a new meaning for the latter. But meaning has already been produced for 'X+4b=Y'; why would a pupil take aboard this new meaning instead of, or even in addition to, the original one, naturally produced by taking 'X+4b=Y' directly in relation to the kernel? There seems to be a didactical paradox here.

Before we set out to solve those paradoxes, a key question must be answered: why would *we* want to introduce a new way of producing meaning, when the previous one seems so promising? The answer to this question is as simple as it is crucial in solving the paradoxes: we do this because *we want* pupils to be able to operate both ways: *we want* them to be able to produce meaning linking a statement to a kernel and *we want* them to be able to produce meaning by performing direct transformations of statements. But this is essentially a decision made on a cultural basis: that is what our culture expects from someone who is performing the function we are; there is no plausible reason to believe *a priori* that in any other given culture people are expected to be able to produce meaning through the direct transformation of statements.

What makes this a key assumption in the solution of the paradoxes, is precisely that our function as interlocutors will provide the reference for the intention to constitute whole statements into objects, at the same time we, as interlocutors, are the agents of trying to get the pupils to engage in the activity of producing meaning in the new way we are proposing.

The paradoxes are solved, then, by observing that there is an intermediate step in which the *intention* to engage in a new activity is the key factor, and during which — however brief it is — authority plays a crucial role. It is never too much to observe that I am using the notion of authority just as to indicate a reliable point of reference.

The statements are, then, first constituted into 'objects whose properties I do not know (although I know they exist because my reliable interlocutor indicates so).' This is not very different from pupils listening to a lesson about a distant country: 'There is a place where people…', and that is precisely what constitutes the country and the people the pupils engage in thinking about. It is certainly essential that they have already produced some meaning for 'people' and 'country.'

Both epistemological and didactical paradoxes are solved at once: the first meaning for a statement as a whole is precisely 'statements can be treated as a whole', and the justification is the teacher's authority, although probably nothing else is at first known about *what* really can be made with them; on the other hand, pupils engage in that activity — *if they do* — because the teacher represents — if s/he does — the legitimacy of the newly proposed way of producing meaning, and because pupils want to belong to a social practice in which that way of producing meaning is legitimate and desireable. The paradoxes were rooted, in fact, in conceiving the possibility of a transition from the old to the new. But the formulation I present makes clear that it is not the case of a transition, but actually it is the case of a rupture, and that the rupture is promoted within a process of interlocution: 'let's do it differently,' and someone has to have a reason for doing it differently.

The next step would be the production of new correct statements in relation to the tanks situation, but with the requirement that two justifications are produced: one in relation to the kernel and another produced by a direct transformation of a previously accepted statement. The reason for this is precisely to make explicit the existence of two different ways of producing meaning for new statements.

It is possible to produce meaning for the statement,

S5 'X-1b=Y-5b'

both with

J5A 'If I take 1 bucket from X, 10 buckets will be missing; the same happens if I take 5 buckets from Y, because 5 buckets were already missing, and I am taking another 5.'

and with

J5B 'Take 5b from each side of X+4b=Y, which I had already established as correct.'

As soon as the property 'one can take the same from both sides of a statement' is established, it is possible to produce statements like 'X-40b=Y-44b,' which although correct according to the properties of the objects 'statements,' cannot be made meaningful within a Semantic Field of tanks, simply because it does not seem plausible that one can remove 40 buckets from X.

It is now possible to produce a strong distinction between the two modes of producing meaning, on the basis of the fact that there are objects in one case which cannot be made into objects in the other. The crucial step is, I insist, to produce a rupture, not a transition.[9]

There are several possibilities to follow from here. One of them is to start working on producing 'target-statements' from statements already produced, for instance, from 'X - 1b=Y-5b' to produce a statement of the form 'X=...' or 'Y=...' or, later, of the form 'b=...'. In one of the groups, two solutions appeared to the 'target-statement' 'b=...' from 'X+4b=Y': 'b=Y-X-3b' and 'b = $\frac{X-Y}{4}$' The latter came from a student whose mother is a mathematician, who gave her 'a hint.'

The activity described above is intended to provide a design model. On the basis of the rationale for that design is the fact that meaning can be produced for algebra in a number of different ways. It is meant to indicate that the so-called 'concrete embodiments,' if taken together with the assumption of the possibility of a transition — the traditional didactical effort to produce a 'silent transition' — is likely to hide from pupils the fact that they are being required to produce meaning within a new, and distinct, Semantic Field. Moreover, it indicates how it is possible to overcome such problems and still keep the possibility of starting from already constituted, familiar, kernels.

Three axis were taken into consideration on the design proposed: (i) one which goes from *solution-driven* to *method-driven* activities, the latter being characteristic of the Tanks activity; (ii) language, representations and notations; and, (iii) the logics of the operations, that is, kernels and Semantic Fields.

SUMMARY AND CONCLUSION

One key thing has been shown in Parts 1 and 2: that the characterisation of knowledge adopted by traditional epistemologies is inadequate, in particular for mathematics education. As an alternative, I have presented a characterisation of knowledge which incorporates — as a constitutive element — the justification a person has for believing that something is the case.

The notion of knowledge as a pair (statement-belief, justification) is the basis for the construction of the Theoretical Model of Semantic Fields. Within that model, the production of meaning is an activity which happens around kernels constituted by local stipulations. That activity constitutes Semantic Fields. Justifications have the double role of constituting objects and of taking part in the process of a person being granted with the right of knowing that such and such is the case. Objects are, then, constituted within Semantic Fields.

Whenever objects are constituted, there is a particular logic of operations which applies to them, i.e. what can be done with them.

Knowledge is always enunciated to an interlocutor. Within the Theoretical Model of Semantic Fields, interlocutors are an essential part of cognition, as the production of meaning is always directed towards an interlocutor. When we produce meaning we are speaking to an interlocutor, either internal or external.

Characterising algebra as a text, rather than as knowledge, allowed us to account positively for different meanings produced for it, without having to slide into a hierarchy in which 'official' ways of producing meaning are at the top. There are two key consequences: (i) we are not forced any longer to treat children's cultures nor any other culture as 'lacking'; and, (ii) we are able to characterise the process by which meaning production might — if not properly dealt with — constitute limits for pupils' learning. With respect to item (i), it is important to point out that any epistemology which characterises what *is* on the basis of the very culture within which it has been produced, is clearly unable to be of much use in helping us to move forward, to go beyond limits historically and materially produced.

A number of educational consequences can be drawn.

First, that instead of simply looking for 'meaningful learning', we must take into account the possibility of different meanings, and that we may be particularly interested in getting pupils to produce meaning in a specific way; in the case of algebra, the various ways of producing meaning are of interest, but we may be particularly aiming at getting them to think algebraically, although not to the exclusion of other possibilities.

Second, it is possible to organise classroom activity around different modes of thinking, rather than in terms of 'content' as given in mathematics itself. One key aspect of this new organisation, is that contents would be *naturally* integrated. For instance, around 'thinking with wholes and parts' one would have fractions, some algebra, some geometry. Around 'thinking with scale-balances' one would have some physics, some algebra, some measurement. I am not mentioning the possibility that contents from outside mathematics are also taken aboard. In a sense, organising classroom activity around modes of thinking — as I characterise them — has some flavour of project-oriented approaches; I believe the two ideas can and should be thought of together.

Third, that by getting pupils to make justifications explicit we may go far beyond the simple possibility of checking whether they 'really' know what they are saying. Through this process pupils and teacher will be able to produced *shared* meaning and *shared* knowledge. In terms of the teaching process, that enables the teacher to identify and approach, from inside, situations where learning is not occuring. Whenever the teacher has to deal with it only from the outside, there are two possibilities: (i) insist on the approaches already used, as if pupils needed a second chance to 'see' what they did not in the first try; or, (ii) leave it to the pupil, perhaps in the sense of assuming that the pupils was not yet ready to learn those ideas being proposed. In both cases the teacher is very much passive. But by entering into the pupils' world of meanings, and by making explicit that at some points new ways of producing meaning are being proposed, both teacher and pupils become truly active in the constitution of a common, shared discourse. The sharing of statement-beliefs *and* justifications, on the other hand, are not seen any longer only as a politically correct attitude, in the solidarity sense of sharing; that is certainly important, but there is now also the fact that such a process is an essential, constitutive part of learning, as it is through this that the legitimacy of given modes of thinking is eventually established for those 'listening'. I had already mentioned the role of the teacher as an interlocutor; the role played by pupils among themselves is quite similar to that of the teacher.

Fourth, and last, there should be a shift away from the usual 'concrete to abstract' notion. The suggestion is that we stop thinking of scale-balances and function machines, for instance, only as intermediate steps in the road towards 'what algebra really is.' Instead of asking the question 'how to bridge the gap,' perhaps we would rather acknowledge that there is no possible 'bridge,' there is no transition. The idea of a transition is certainly rooted in the notion that there is a higher level of thinking to be reached from lower ones. Davydov's work, and my own, shows that changing the perspective with respect to 'concrete to abstract' allows us to produce powerful classroom approaches.

Although widely aimed as a model for epistemology, I think that the Theoretical Model of Semantic Fields provides a simple, yet powerful, tool for research and development in mathematics education, as well as for guiding classroom practices and for enabling teachers to produce a sufficiently fine, thus useful, reading of the process of meaning production in the classroom. Finally, I would like to emphasise that the Theoretical Model of the Semantic Fields is not a 'local theory' aimed only at the production of meaning for algebra; it is similarly applicable to all parts of Mathematics. In fact, it applies to any process where the production of meaning occurs, but this is certainly not the place for such a discussion.

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NOTES

1 It is sufficient to say, at this point, that an object for a person is anything which that person can say something about. Strictly speaking, '3x+10=100' might be constituted into objects even before any 'mathematical' meaning is produced, once the person may say that '3x+10=100' can, for instance, be typewritten or handwritten, and in large, medium or small size. This is, however, a subtle aspect which will not be discussed any further here.

2 In Lins (1992), I have characterised Algebraic Thinking as thinking arithmetically, internally, and analytically. Thinking arithmetically can be understood as 'thinking in numbers'; thinking internally means not modelling back those numbers as some other objects (eg, measures or wooden sticks, currency or areas), ie, characterising them only as objects having given properties in relation to the operations and to equality and eventually inequality; and, thinking analytically means treating unknown numbers exactly as if they were known. Put together, these three conditions point to objects, numbers, which are known only as objects we operate on with the arithmetical operations.

3 As suggested before, by a text, from here on, I will mean not only written text, but any residue of an enunciation: sounds (residues of utterances), drawings and diagrams, gestures and all sorts of body signs. What makes a text what it is, is the reader's belief that it is indeed a residue of an enunciation, that is, a text is framed by the reader; also, it is always framed as such in the context of a demand that meaning be produced for it.

4 For instance: it may seem natural for us to place equations and functions close to each other, but this possibility depends on constituting them as objects with certain common features; such a constitution may not be, however, within the horizon of a given person's ways of producing meaning for those objects.

5 In the work of the Dutch group of Utrecht (see, for instance, van Reeuwijk, 1995), we find the notions of vertical and horizontal mathematisation. Although similar, I would like to point out that vertical and horizontal developments within the TMSF are much more general notions than their Dutch counterparts, particularly as they are not aimed only at mathematical meaning. In particular, the Dutch

version characterises 'mathematisation' in terms of notation, but without clarifying whether or not this carries with it—within their model—'meaning.'

6 This example is given assuming that meaning had first been produced within a Semantic Field of a scale-balance. A metaphor establishes the first Semantic Field as ontologically more primitive, while the reversed metaphor produces a restructuring of the ontological building: 'now I know that in fact…'

7 Parts of this section have already been reported at PME XIX, Recife (Brasil), 1995. I am particularly indebted to Geraldo Garcia Duarte Jr., Rosamund Sutherland and Luciano Meira, for their insightful comments on the interviews discussed here.

8 Producing meaning for algebra: a research and development project in teaching and learning, a cooperation project conducted under the direction of Rosamund Sutherland (University of Bristol, UK) and the author (Dept. of Mathematics, UNESP-Rio Claro, Brazil); the project is partially funded by CNPq (Brazil), grant nº 530230/93-3, and by the British Council.

9 An illuminating instance of the need of familiar kernels is found in the use of 'examples.' For instance, when I teach Group Theory, it is common that when presented with the definition of a Group my students are completely unable to say a word. After discussing a few examples, however, many of them become able to produce statements justified within a Semantic Field of the formal definition. Familiarity with the examples, here, allows them to say things.

10 On a more technical note, it is worth indicating a general mechanism of production of new Semantic Fields, namely, the introduction of a new operation on objects previously constituted. In the history of mathematics we find a prime example of that mechanism in action, when Wessel introduces a multiplication of directed lines, objects which are first constituted as displacements (Wessel, 1959), and, as a consequence, produces new objects, which are distinct from the previous ones; in this particular case, Wessel then associates these new objects with complex numbers, by showing that the addition and multiplication of directed lines have the same properties as those of complex numbers. It is interesting that he is not seeking a foundational model for complex numbers; instead, he is trying to develop a 'geometric calculus'—after all, he was a surveyor—, and what he shows is that complex numbers are helpful in dealing analytically with directions, that being his main objective.