ON THE PRODUCTION OF MEANING FOR THE NOTION OF LINEAR TRANSFORMATION IN LINEAR ALGEBRA: KIKA AND VIVIAN SPEAK[1]

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ABSTRACT

In our study, based on the Theoretical Model of Semantic Fields (Lins, 2001), we analysed the production of meaning for 'linear transformation' (LT), aiming at producing elements to support a further reflection on the teaching and the learning of Linear Algebra. As part of the study we have conducted interviews with two students of a first Linear Algebra course (undergraduate mathematics degree), seeking to elicit the meanings they were producing for that notion while engaged in trying to 'talk about' particular (and non-usual for them) LT's presented to them. Two of the aspects considered in the analysis were the meanings being produced (and the kernels thus involved, see Lins 2001) and the texts being produced (notations, diagrams, writing, speech, gestures). For instance, we have found out that the students always tried to find a way to visualise the LT's in question (as one may visualise the usual R^2 as a geometric plane). This study is part of a broader project ('A framework for the mathematics-content courses in the university preparation of mathematics teachers') and aimed at producing elements that allow an adequate reading of the process of meaning production in the classroom, leading to new approaches to deal with students' difficulties and to new approaches to the classroom practices of mathematics professors engaged in mathematics teacher education.

KEYWORDS: linear transformation, linear algebra, meaning production, semantic fields

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Introduction

The study reported here is a section of a larger study on the production of meaning for the notion of linear transformation in Linear Algebra. In this section of the study we wanted to examine the meanings two undergraduate students would produce for linear transformation in specific situations.

Other sections included a study of mathematical texts (historical and present-day) in which it is possible identify ideas related to our present characterisation of linear transformation, for instance in the work of Vieta and in the work of Peano.

The theoretical support for the study comes from the Theoretical Model of Semantic Fields (TMSF; see, for instance, Lins, 2001). Its central notions are those of ‘knowledge’ and ‘meaning’. ‘Knowledge’ is characterised as a statement in which a person believes (a statement-belief), together with a justification s/he has for making that statement. ‘Meaning’ is characterised as what a person actually says about an object, in a given situation (activity); but it is *not* everything that a person *could* eventually say about that object. Those two notions are naturally useful for producing a dynamical reading of what people are thinking in given situations, a reading of processes rather then of states.

Our primary interest here was not to identify patterns of thought (in terms of its content) which would be immediately generalisable and, in this sense, it was not relevant that the students chosen be in any sense 'typical'. Rather, we wanted to elicit the extent to which it would be possible, with the support of the TMSF, to identify the meanings being produced by particular students in particular activities, for the notion of linear transformation; for this reason, a larger sample was not required. Nevertheless, it is clear that a larger scale study, conducted with the support of the TMSF and in a way similar to the present one could reveal more generalisable patterns; in this sense we think this study suggests a possibly fruitful line for future research. In this direction, the data gathered in this study is consistent with data gathered in other studies of our larger research project, and it suggests that natural and naturalised objects (such as a 'natural' notion of 'space') might play a central role in the production of meaning for mathematical objects and also that they are quite 'resistant' to the usual mathematics courses at university.

What we understand as generalisable, coming from this study, is the approach to the reading of processes of meaning production, which we adopted, and its usefulness in revealing details that would otherwise be missed. Based on what we have learned through the interviews we can suggest that such an approach is indeed required in the classroom if one wants to interact productively with students and if one wants to organise teaching as to be effective. With respect to the latter, our particular interest is in the mathematical education of mathematics teachers.

In the case of the students we interviewed, it is safe to say that the meanings they produced for linear transformations were not in line with what professors teaching Linear Algebra expect the students to produce, particularly in the sense that the notions of (vector) space, vector and linear transformation remained strongly linked to natural notions of space (as the physical space), vectors (as arrows) and linear transformations as operators that 'change' vectors into vectors of the same kind.

This paper focuses on four of a set of five interviews conducted with each of two undergraduate mathematics students. The interviews were designed with different purposes but in all cases there was the intention of getting them to speak as much as possible, to tell us as much as possible, in an explicit way, of their ideas and understandings. Our interest was on what the objects they were talking about actually were *for them* in those specific situations.

The interviews: Kika and Vivian speak

A set of five interviews were conducted separately with each of two mathematics undergraduate students, Kika and Vivian. At the time of the first four interviews they were taking a course called 'Introduction to Linear Algebra', focusing mostly on $\mathbf{R}^2$ and $\mathbf{R}^3$ with the usual structure; at the time of the fifth interview they had already finished the introductory course but had also successfully finished a second course on Linear Algebra which focused on abstract vector spaces.

Interview 1 was set to elicit what they would spontaneously say about some notions from Linear Algebra. Interview 2 was set to elicit some of the relations they established between these notions. Interview 3 was set to elicit how they would talk about a given linear transformation (given by the explicit transformation rule). Interview 4 will not be discussed in this paper. Interview 5 was set to elicit the extent to which their understanding of linear transformations depended on visualisation and on 'the natural space'. A delay between interview 5 and the others (11 months) allowed us to examine the extent to which visual-geometrical meanings had 'resisted' to the work with non-visualisable, abstract vector spaces.

Our observations will 'track' what Kika and Vivian said related to 'linear transformation'.

INTERVIEW 1

Interview 1 consisted of presenting them with a list of notions from Linear Algebra (matrix, linear transformation, sets of linear equations, vector space and vector), asking them to write down what they had to say about them and we subsequently spoke with them (separately) about what they had written.

With respect to 'linear transformation', there was a marked difference between what Kika and Vivian said. Although both mentioned it is a mapping with two special properties, while Kika gave this as the only characterisation, Vivian seemed to associate linear transformations with 'transforming', in the sense of doing something to the vectors:

> [VIVIAN] "Linear transformation I think is a mapping that has some *transformations*, like, like the rotation." (our emphasis)

This initial impression was later confirmed by the other interviews, when Kika too spoke of mappings as 'acting' on vectors and 'doing something'.

A particular aspect of what they said was quite important to us. Both of them referred to 'vector space' as being,

> [VIVIAN] "…all places where the vectors live, like, where they act. And also you can multiply a vector by a real number and it remains in this same, this same little place there, where they live, this same little house."

> [KIKA] "…the space where the vectors act, where we operate with them."

This natural notion of space as 'a place' will be present in all interviews and this has an important consequence. For both of them it is crucial that it be possible to *visualise* the space where the vectors are to be and this has to do with visualising the vectors; without this a vector space does not make sense and talking about a linear transformation involving such space becomes talking about a mapping only, as we will show on Interview 5, when they are faced with a space of matrices. Coherent with this, both understood vectors as [KIKA] "…an oriented line segment… [also used for] representing speed…".

At this point they were dealing only with $\mathbf{R}^2$ and $\mathbf{R}^3$ in the introductory course and it would be reasonable to associate them with the physical space around us but, as we will show, those understandings 'resisted' the second course they took, involving spaces and vectors which could not be easily or at all visualised that way. In the final section we discuss a possible implication of this for teacher education.

INTERVIEW 2

The following 'names' were each written on a card: matrix, basis, set of linear equations, determinant, linear transformation, vectors, linear combination, system of generators, dimension, mapping, vector space and linear independence. Nineteen random draws of three cards each time (the same draws for both students) were made and the students asked to group for each draw the two they saw as more closely related. Then each student was interviewed, about their choices.

There are two remarkable aspects in the groupings, particularly because in all these cases the answers of Kika and Vivian coincide. First, that every time 'linear transformation' and 'mapping' were on the same draw they were grouped together. Second, that every time 'linear transformation' and 'vector space' were on the same draw they were *not* grouped together; in the two draws containing both but not 'mapping', each was left out once.

The combined suggestion is that they understood a linear combination as a mapping only and took the operations in a naturalised way; being simply "places where the vectors live", vector spaces were not part of their understanding of linear transformations. That can be seen when they are asked to group 'vector space', 'linearly independent vectors' and 'linear transformation'. Kika groups the first two and excludes 'linear transformation', and explains:

> [KIKA] "The LI vectors live isn't it in the vector space and the transformation acts on the vectors of the vector space. But first they would be there and then it would act."

Vivian has a similar explanation for the same choice:

> [VIVIAN] "Because… the vector space is like the vectors' little house and then [it] has to be together with the vectors."

It is interesting to notice that in Kika's statement the fact that the vectors are linearly independent has no relevance, although she mentions it, and that Vivian does not even mention it.

When the draw was 'vector space', 'set of linear equation' and 'linear transformation' both grouped the last two and excluded the first; here the reason is probably straightforward, as their professor (following Banchoff's book) had defined linear transformations as given by a set of linear equations.

Summarising the relevant insights from this interview, it seems that those students had a natural understanding of 'space' ('space' as the space we live in, even if presented as $\mathbf{R}^2$ and $\mathbf{R}^3$) and that the teaching had not addressed this fact properly. That given, it was coherent in their thinking that linear transformations were seen as mappings 'only', particularly in the sense that 'mappings do something', as one finds in school mathematics.

INTERVIEW 3

Interview 3 consisted of presenting each student, separately, with a sheet with the question:

> How would you describe the transformation $T(x, y) = (-y, x)$ from $\mathbf{R}^2$ into $\mathbf{R}^2$?

They were allowed to think about the question for about 20 minutes and asked to write down their ideas. After that each student was asked to go to the blackboard and present her conclusions to the interviewer.

In both cases the first statement is that T is a linear transformation. But while Kika actually begins with the 'calculations' (as they called the algebraic verification of the properties) to verify that T is a linear transformation, Vivian never writes or says anything that shows she had actually done them.

The key to understanding Vivian's thinking seems to be in her answer when asked why she thinks that T is a linear transformation:

> [VIVIAN] "Because it satisfies the properties of linear transformation, both [properties]. *And it's called a reflection, this is going to be a reflection on the x-axis. I will draw.*" (our emphasis)

Somehow she convinced herself of the underlined statement (which is not correct) and our interpretation is that assuming that reflections are linear transformations she got to the conclusion on the first statement. in fact all her interview is clearly dominated by making drawings and talking about what the mapping 'does':

> [VIVIAN] (making drawings on the blackboard) "Let's give an example. I'll take a little vector here. Let's suppose with coordinates x, y, any. Here. By the transformation T it'll be *taken* here […] it will be simmetrical in relation to the x-axis. The angle here is the same, everything simmetrical. This size here, this lenght, will be the same as this [comparing the original vector and the image]. And, yes, it is a linear transformation."

and after realising she was mistaken about T being a reflection on the x-axis,

> [VIVIAN] "Let me think something. [silence] *A linear transformation is something*" (our emphasis)
>
> […]
>
> [VIVIAN] "Going back. It's a rotation."
>
> […]
>
> [VIVIAN] "It's a rotation. It's a rotation of pi over two […]"

Although stating she had done the calculations, they *never* materialise in *any* form (and she gets terribly messed up when trying to talk about them); all the time it is clear she is trying to determine 'which' transformation T is (among the 'prototypes' available to her) and *that* is what will tell her whether T is or not a linear transformation.

Kika, on the other hand, went straight on to the calculations (for the properties) and continued to show that T is injective (by showing that ker T = {(0,0)}). She then considered geometrical aspects (lenght preserving, angle preserving, by drawing particular vectors and their images under T), applied T to the canonical basis and determined the matrix associated to T. Only then, looking at the matrix, she decided T was a rotation of 90^{o} (as in the original written protocol).

Two aspects are more relevant in these interviews. First, the clear difference in the meanings produced by the two students for 'linear transformation' and the consequences of this on the way they deal with the task. Kika is dealing with a mapping that may or may not have some given properties (including being injective), while Vivian is dealing with a mapping which does something to the vectors and it is this 'something' that is central in characterising the mapping.

Second, that Vivian's thinking seems based (again) on a naturalised notion of space, while Kika's seems much less dependent on that. But after interview 5 it became clear that Kika's thinking here was, in fact, quite particular to the task, as the relation between $\mathbf{R}^2$ and the naturalised space was not problematic, that is, the effect of thinking with a naturalised space was not visible.

INTERVIEW 5

Interview 5 had two questions. Each of them was presented to the student on the blackboard and whatever they wished to write or sketch had to be done directly on it. One question was presented first and then discussed; when the researchers were satisfied with the discussion the second one was presented and discussed:

1) How would you describe the mapping

$$f : \{ax + b; a, b \in \Re\} \to \left\{ \begin{bmatrix} a & b \\ 0 & -a \end{bmatrix}; a, b \in \Re \right\}$$

given by

$$ax + b \mapsto \begin{bmatrix} -b & a \\ 0 & b \end{bmatrix}$$

and

2) How would you describe the mapping

$$g : Z_5^2 \longrightarrow Z_5^2$$

given by

$$(x, y) \mapsto (-y, x)$$

These interviews were, in at least three aspects, quite different from the previous ones: (i) the vector spaces involved were not $\mathbf{R}^2$ or $\mathbf{R}^3$; (ii) in one question the mapping was not an operator and in the other the field was finite; (iii) at this time the students had already taken with success a second course on Linear Algebra, after the introductory one. As we had already said, the interview was set to examine the extent to which their understanding of linear transformations depended on visualisation and on 'the natural space', that is the extent to which visual-geometrical meanings had 'resisted' to the work with non-visualisable, abstract vector spaces.

Because of the limit imposed on the size of this paper, we will focus our attention on one student (Kika) and on the first question; we chose Kika here because on interview 3 she had preferred to verify the linearity algebraically (something necessary on both questions of interview 5), rather than visually as Vivian did. Further ahead we briefly comment on the other student and on the other question.

When asked about it, Kika stated that ***f*** is a linear transformation and sketched the calculations to support her statement. However, she is greatly disturbed by the fact that she cannot 'see' what the space of matrices is:

> [KIKA] "…It's difficult to talk [about ***f***] because it is a pretty weird mapping."

> […]
>
> [KIKA] "…because it takes a straight line [sic] into a matrix so, it is not something you can describe too clearly. […] There is no way to describe the *form* of a space of matrices […] when I think of mappings I always think of [the] domain [being] the real [numbers] […] it can take to the space of the real [numbers], can take to a circle. I imagine like how the image set would be, right, how I would be describing it and quite frankly in this case I don't know how to describe its *image*." (our emphasis)

'Image' here is clearly used by Kika in a visual way.

To make visual sense of the domain she immediately described its elements as "straight lines" and later said, explaining the calculations she had done to show that ***f*** is a linear transformation:

> [KIKA] "…if you take two distinct straight lines and you say they take you [sic] to a matrix of this form and if you apply ***f*** first to each of them and add the matrices, add the images, I think it would be the same thing if I take the two straight lines, put them together into a single straight line and apply the function. So in this case I think it's right…"

It could seem that she is only using the *name* 'straight line' given the strong link she establishes between the polynomials and the associated polynomial functions. But when she talks about her understanding of the domain there is little doubt that this is not the case:

> [KIKA] "The domain is all the straight lines, right, it would be R two. […] Because here it would be the equations of the lines."
>
> […]
>
> [KIKA] "…as I vary the *a* and the *b* over all the real [numbers] I will be getting distinct lines, like this, this, this [drawing lines on a diagram with two orthogonal axis].… All ways. So they will occupy the whole plane, all R two. That's how I'm thinking. Because I can vary them over all of R, then it would be the whole of R two. All the sets [sic] of possible straight lines […] Easier to *imagine* than the matrices." (our emphsais)

We think there is a strong suggestion here that Kika thinks vectors must have a visual image that somehow corresponds to arrows (oriented line segments) and that this is related to a naturalised space. She finds a way with the polynomials, but once she cannot produce such a visual image for the space of matrices, the mapping is treated only as a mapping and not as a mapping between vector spaces (she talks about ***f*** being or not injective and surjective, and uses the expression 'homomorphism', which does not belong to typical Linear Algebra courses in Brazsil). When one of the interviwers presents her with the statement "this mapping will take a given straight line of the first space to a straight line of the second space", Kika gets in trouble; after an exchange on what could straight lines be on the second space she said:

> [KIKA] "Only if I took that matrix and multiplied it by a vector, I don't know, x, y, and it would be like that, the matricial multiplication…"

That indicates, we think, the extent to which she was not able to produce meaning for the statement presented to her, looking very much as a desperate attempt to make any sense of what had been presented to her. When working on question 2 she said that she could not think of rotations unless she could see the straight line that was being rotated and actually laughed when she was told it was a rotation of 90^{o}:

> [INTERVIEWER 2] "I will tell you: that is a 90^{o} angle. Do you want me to prove that the cosine of that angle is zero?"
>
> [KIKA] (laughs)
>
> [INTERVIEWER 2] "Do you?"
>
> [KIKA] (laughs)
>
> […]
>
> [INTERVIEWER 2] "If I proved to you in a way you could accept…"
>
> [KIKA] "…see…"

She actually *corrects* the interviewer to say that what she needs is not convincing, is *seeing*.

On the first question Vivian, the other student, had a similar difficulty in accepting the space of matrices because she could not visualise them as 'vectors'.

On the second question both students represented the set of vectors as points on a Cartesian diagram, so they could *see* what the transformation 'does', and although in interview 3 both ended up talking of a transformation with the same rule [$(x, y) \mapsto (-y, x)$] as being a rotation, they did not do so here. In interview 3 Kika had verified algebraically that the transformation was linear, but not here, suggesting that on question 2 of the fifth interview visualising the vectors was necessary *before* it made sense to engage with the algebra. Vivian, as she had done on interview 3, depended almost completely on what she could see the transformation 'doing'.

Conclusions and implications for the classroom

Overall we think that the data gathered through the interviews suggests that there might be a huge gap between successfully taking the two Linear Algebra courses they took and developing a mathematically sound understanding of the objects of Linear Algebra, particularly those of vector space (as a structure), of vectors (as elements of the base set of a vector space) and of linear transformation (as a homomorphism between vector spaces, which are structures). It also shows that the role of a naturalised space remained, in their case, considerably untouched by the ideas discussed during the courses.

The metaphor we have been using to describe this situation is that as the students go into the classroom they leave their natural ideas outside, then try their best to succeed inside the classroom and as they leave the classroom they leave the mathematical ideas inside, take the natural ideas back and go home. Instead of saying that as people finish their schooling they forget the mathematics (for instance, at the end of high-school), we say they do this *everyday*.

The mathematical education of mathematics teachers is the main object of our larger research project, and we strongly suggest that the situation above is highly undesirable in this case, for two reasons. First, because almost all the benefit for her/his mathematical development is reduced to practicing bits of school mathematics that appear during the 'advanced mathematics' courses, for instance, calculating with matrices or doing some analytical geometry, in the case of Linear Algebra.

Second, and more harmful, because the future teacher does not develop an awareness of the process described in our metaphor and for that reason s/he is unable to become capable of dealing with this situation in her/his professional life. To promote such an awareness is what we call 'to educate *through* mathematics' and we consider it to be a key component of the mathematical education of mathematics teachers.

On the basis of our analysis of the interviews there were two questions: (i) which are the objects the students are thinking about?; and, (ii) what are they saying about those objects? Technically speaking, those two questions must be seen as one (Lins, 2001).

We focused our analysis on three objects constituted by Kika and Vivian: space, vector and mapping/function/linear transformation. A naturalised space preceeded the others, as the place where things are, can be, and that does not depend of anything else for it to be conceived of, much like the physical space. Then there were vectors as arrow-like objects, which only made sense as visualisable (inside some naturalised space). Finally, there were functions which almost only made sense as literally transforming a vector into another vector *of the same kind* (operators), as if a vector itself was streched or rotated 'by hand' (again a naturalised space and naturalised operations underlying this possibility).

Those understandings are coherent among them and, as far as we could probe, they resisted to many hours of talk on abstract vector spaces and their properties; moreover, without assuming—at adequate times—a different understanding of those notions, much or all of Linear Algebra is quite useless in the education of future teachers.

Also, the approach we adopted to read the meanings being produced by Kika and Vivian proved to be quite adequate and useful, as it allowed us to go much beyond simply stating that they did not 'know' what a linear transformation 'is', that 'actually' they had not learned. It allowed us to produce a positive understanding of their thinking which showed a consistent set of objects (space, vector, transformation) at the kernel of their thinking, and allowed us to understand how their actual thinking did not correspond to what their professors likely expected from them. A crucial question to raise here is how they could be successful at the two courses they took on Linear Algebra.

Evidence such as that presented in this paper suggest the need for developing teaching approaches that treat natural ideas explicitly, by giving students a chance to talk about them, discussing them in their relation to abstract, non-natural, ideas. In the particular case of the education of mathematics teachers we believe this would require major changes in the way it has been traditionally conceived (mathematics courses plus pedagogical complementation).

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