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Chapter 4

The Early Development of Algebraic Reasoning: The Current State of the Field

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Abstract: The main aim of this chapter is to argue that an early start to algebra education is possible and of great relevance for mathematics education because it provides a special opportunity to foster a particular kind of generality in our students' thinking. To argue this, we map the various views on algebra education found historically, and trace how the perceptions that mathematics educators hold about children's thinking and learning have changed. Overall, a great realisation that children can do more in mathematics than was previously believed leads to the adoption of more ambitious objectives for the initial years of school, and to the development of new classroom approaches to algebra education in the early grades. That does not mean teaching the *same old school algebra* in the same usual way to younger children, but rather to introduce them to new algebraic ways of thinking and immersing them in the *culture of algebra.* The chapter ends with a research agenda to further developments in this particular sub-field of mathematics education.

Key words: Early algebra, algebraic thinking, generalised arithmetic, symbolic arithmetic

4.1 Introduction

In this chapter, we attempt to provide the basis for an understanding of the early development of algebraic reasoning and the larger views of algebra education in which this may occur. Our intent is to help *algebra educators* move forward in the task of creating new approaches to algebra education that incorporate both the practices of the past that proved fruitful and the new possibilities offered both by the available technology and by recent views of cognition and learning. To achieve this, we step back to analyse the expression *algebra education*, in order to move beyond a content-centred characterisation.

Defining *algebra* is fraught with difficulty, especially if one expects tight and closed epistemological definitions, because what one takes to be algebra depends on many cultural and other factors that vary widely across and even within communities. Some of this variation is reviewed in Chapter 13 of this book. Nonetheless, we have been able to agree provisionally on two key characteristics of *algebraic thinking*. First, it involves acts of deliberate generalisation and expression of generality. Second, it involves, usually as a separate endeavour, reasoning based on the *forms* of syntactically-structured generalisations, including syntactically and semantically guided actions. This is a characterisation of the broad kinds of algebraic reasoning that helps us then discuss forms of algebraic thinking appropriate for young children and the conditions that may promote them. Among such conditions, for instance, is a need for greater integration of different mathematical topics, in order to promote the development of algebraic forms of thinking, which would yield better problem-solving abilities in students. Another consideration is the recognition that algebraic thinking empowers students by providing tools that allow a great degree of certain types of generality, something that has, of course, been taken to be true for a long time, but this time considering the empowerment of much younger students than usual.

The theme of this chapter, *early algebra*, allows us to look both sideways (how to integrate algebra education with other topics at all levels of schooling) and ahead (the implications of what is done *early* for the following grades). In other words, instead of algebra education being restricted to a more narrowly defined grade band or narrow sequence of courses or learning environments, we can examine the possibility of creating a *new algebra world* from the beginning. Because of this, Section 4.3 (about the implications) and Section 4.4 (a research agenda), assume a quite important role in this chapter, as it is there that we argue how the suggestions and indications of the more specific discussion on early algebra could become part of the bigger picture.

Two understandings of what *early algebra* means now seem to be current. The first, and for many years the more ubiquitous, refers to the first time students meet algebra in school. For many different reasons—sometimes tradition, sometimes dominant theoretical positions, sometimes the impact of published studies—that first encounter was likely to happen when students were about 12-13 years old, in some cases even older. This first understanding of early algebra applies to most of the other work reported in this book, including most but not all of the discussion on approaches to algebra (see Chapter 5). The second understanding, which only slowly and more recently has been gaining ground in the mathematics education community, takes *early algebra* to refer to the introduction of students to algebraic reasoning at a much earlier age, sometimes as young as seven years old. The approach we take in this chapter is to focus on fostering the development of *algebraic thinking*, and not at the teaching and learning of specific bits of algebra content. Whatever content or activity is useful in helping the teacher to achieve that goal might become part of early algebra. It would be impossible, of course, in the body of this chapter, to present the kind of examples that show more clearly how this can be done, so we strongly suggest that the reader take the many references we point to as an important follow-up to this chapter.

We will argue that the increased acceptance of the second view is related to the fact that it is only more recently that the mathematics education community began to realise seriously that younger children could do much more than was previously supposed. The old supposition is a consequence of the already mentioned combination of tradition and dominant theoretical positions acting as constraints and blinders, though we think that other factors are at work sustaining the historically received view of algebra education.

Changes in the views on what is learning and how formal education should be organised to integrate those new views, led to a more enlightened view of the way mathematics educators saw children's work. In coming to be seen as a truly long-term process, algebra education began to incorporate the idea that getting accustomed to particular aspects of algebraic activity (e.g., formulae and literal notation as well as written expressions containing indicated operations) was as relevant as mastering the syntactical structures of traditional formalisms.

Below we will argue that an early start in algebra education is not only possible but is necessary, and will focus on the different forms such early starts might take and the key assumptions that they are based upon.

4.2 Algebra Education in the Past

To understand the significance of the approaches to early algebra proposed in this chapter, it is necessary to examine the developments that preceded their emergence. Those developments may be grouped into three periods. During the first period, tradition ruled unchecked, reigning only for the reason of being tradition, and without support other than experience. The second period saw research begin to investigate the processes underlying the approaches adopted by the traditions of the first period. Finally, during the third period, the view that arithmetic should precede algebra began to be examined.

In this section, our overall aim is to give an overview but not, to any extent, to present a thorough literature review. The papers mentioned were chosen only for their exemplary character with respect to the points being discussed.

4.2.1 Relationship between arithmetic and algebra in school traditions

When we look at school traditions in different countries, the relationship between algebra and arithmetic is almost always characterised as *algebra is generalised arithmetic*. Lee (1997) reminds us that, in 1929, the advice of the British Mathematical Association was that "Historically, algebra grew out of arithmetic and so it ought to grow afresh for each individual" (p. 219). She goes on to say that:

> Chevallard, who undertook the examination of school textbooks over the centuries, certainly confirms that this was the direction taken in the introduction of algebra in schools up until the *new math* reforms of the 1960s. The justification and motivation for algebra lay in the presentation of solutions to some traditional arithmetic problems using the tools of algebra. (p. 211)

It seems safe to say that, even today, the *arithmetic then algebra* tradition persists in most countries, perhaps with a new justification added, namely, that (school) algebra is more abstract (and so, more difficult) than arithmetic, which is more concrete (and so, easier). Although some researchers strongly deny that claim (for example, Davis (1975, 1984) pointed to the complexity of certain arithmetic operations compared to core algebra activities such as solving linear equations), it is indeed a dominant view, and the reason for this can be found in the strong dominance of Piagetian constructivism. As algebra would require formal thinking, while arithmetic would not, and as formal thinking would correspond to a later developmental stage, algebra should come later than arithmetic (see Petitto (1979) for an explicit analysis of this assertion in relation to a series of teaching experiments). This is a very simplified version of the argumentation, but it contains the essential elements.

The work of Dietmar Küchemann for the Concepts in Secondary Mathematics and Science (CSMS) project, in the late 1970s and early 1980s, combined those two views, the *algebra as generalised arithmetic* and the Piagetian developmental one. On the one hand, although the original book-report from the CSMS survey refers to Küchemann's study under the heading of *Algebra* (Hart, 1984), Küchemann himself (1978, 1984) refers to it as an investigation of children's understanding of generalised arithmetic. On the other hand, the most visible result of Küchemann's work is a reported link between different uses of letters in *generalised arithmetic* and Piaget's levels of intellectual development. The Booth (1984) follow-up study, however, showed that suggestions about student learning made by the CSMS report were unconfirmed. It also indicated that appropriate teaching could eliminate a number of the reported error patterns.

One could argue that the choice of *generalised arithmetic* corresponded to a willingness to separate school algebra from abstract algebra. This would be a reasonable point particularly after the changes in mathematics that happened since the second half of the 19th century. In traditional school algebra, letters always stand for numbers, but in abstract algebra letters can stand for elements from any set where appropriate combining operations have been defined, including permutations, matrices, geometric transformations, and entirely abstract elements. Quite recently Nicholas Balacheff (2001) proposes a distinction between *symbolic arithmetic* and *algebra* in the editors' postscript to the book *Perspectives on School Algebra*.

> The students' solving world will contain symbolic representations (we may call them algebraic) as well as means to manipulate them, but the control structures—all through the solving process—will still refer to the external world of reference attached to the situation by the problem statement. ... Algebra is not there, but instead we see the functioning of what I would call symbolic arithmetic ... the forms of algebraic expressions. (p. 255)

In any case, with the exception of the pioneering work of Davydov and his colleagues in the former Soviet Union (1962, 1975, 1982, 1983), and, to some extent, parts of the work of Dienes (1973), up to the early 1990s practically all the attention of algebra educators was focused either on producing systems of *stages* related to the learning of algebra (developmental or otherwise) or a compendium of difficulties and their sources. We now consider some of that work.

4.2.2 Research on algebra education up to the 1990s

We begin with some exemplary efforts to produce systems of *stages* that could be related to the learning of algebra (although not always exclusively).

Küchemann has already been mentioned as having attempted to link different uses of letters in *generalised arithmetic* to the stages of development in Piaget. Biggs and Collis (1982) proposed the SOLO Taxonomy in which the structure of the observed responses was to be characterised (as uni-structural, multi-structural, relational, and extended abstract responses), rather than characterising the subject or the expected responses. They used algebra items among their examples and the discussion of those examples resonated with the research in algebra education at the time.

Garcia and Piaget (1984) argued that the mechanisms of transition between historical periods are analogous to those found in the transition between psychogenetic stages. They describe one of those mechanisms as the process which produces a succession of three stages: *intra-objectal*, *inter-objectal,* and *trans-objectal* (Lins, 1992). They refer to a history of algebra which they claim to have begun only with Vieta, who lived in France from 1540 to 1603, using as reference their own interpretation of Jakob Klein's 1968 classic, *Greek Mathematical Thought and the Origins of Algebra.* Further details of Vieta's work are given in Chapter 8 and other work on early symbolism is presented in Chapter 9 of this book.

Harper (1987) stops well short of Garcia and Piaget. In a paper discussed widely upon its publication, he attempted to correlate different notational presentations of solutions to problems (rhetorical, syncopated, and symbolic) to cognitive development (Lins, 1992). The notational categorisation is directly borrowed from Nesselmann, who presented it in 1842 with the sole purpose to distinguish the *presentation* of solutions, not their *production* (Heath, 1964).

Sfard (1989) proposed a model for concept formation based on the distinction between two ways in which a mathematical expression can be perceived: as a process (the *operational* aspect) or as a product (the *structural* aspect). Central to her model is the assumption that the operational aspect must *necessarily* precede the structural aspect because it is assumed to be less abstract. Later, at a presentation for the Algebra Working Group of ICME 1992, she returned to the same categories Harper had borrowed from Nesselmann, and used them to characterise the algebra taught at different school levels (cf. Lee, 1997).

These are examples—exemplary, though—of the kind of work done during the 1980s and 1990s, to produce normative systems of stages that could be used to inform algebra education. Although their thinking and methods varied, each of these studies contributed to the assumption that algebra was best left for later in school life.

A second broad group of research papers produced at that time was focused on producing a catalogue of students' difficulties with algebra and the sources of those difficulties. Those difficulties were frequently related to a particular set of proposed stages, in the sense of a misfit between stage of development and teaching, but also to issues related to notation (Becker, 1988; Filloy, 1987; Gallardo & Rojano, 1987; Herscovics, 1989; Kirshner, 1987, 1990; Pereira-Mendoza, 1987) and to difficulties caused by an insufficient understanding of arithmetic (Booth, 1989; Kieran, 1981). This group of papers, typical of the time, shows that the mood among the teachers, so to speak, was rather directed towards trying to improve algebra education by understanding what students were failing to do, either because teaching was out of synchrony with intellectual/cognitive development or because teaching failed take into account what particular students had previously failed to learn/understand in their prior school experience. As much as the effort to produce normative sets of stages, the perspectives taken here led, in almost all cases, to an interest in older students (12 years old and older).

Another set of consistently pessimistic studies examined error patterns in students' syntactical symbol-manipulation work, where it was often the case that this kind of activity was implicitly taken to be the essence of algebra (Lewis, 1981; Matz, 1980, 1982; McArthur, Stasz, & Zmunidzinas, 1990; Sleeman, 1986). This work, as well as work on interpretation of variables (e.g., Wagner, 1981) and reading of algebraic expressions (Wenger, 1987), repeatedly illustrated the fragility and superficial nature of student competence in operations on algebraic symbols and their interpretation.

To summarise: up to the early 1990s research in algebra education was focused on the *sad stories*, on what children could *not* do, rather than on ways to explore what they *could* do, and ways to tap the potential for development.

4.2.3 Research on algebra education opens the way for early algebra

If up to the early 1990s there were mainly sad stories, from then on things slowly began to change. In this sub-section we offer examples of *happy stories* reflecting a shift towards optimism in algebra education research in relation to what children could do.

There are three basic types of happy stories here. First, there is research that suggested directly that younger children could do more in mathematics than previously thought, particularly when provided with appropriate experiences and instruction. Second, there is research reporting changes in the perspectives on algebra education and algebraic thinking and third, research advancing the idea of using new technologies in algebra education. The first group is clearly related to early algebra as proposed in this chapter, in a sense that will soon become clearer. The second group refers to work that helped broaden the focus of research on algebra education, making it possible for subsequent work to have a more flexible view of it, and so making early algebra more acceptable. Quite frequently this research brought areas such as linguistics, history of mathematics, and epistemology closer to mathematics and developmental psychology, which dominated previous research on algebra education. On the third group we will comment later.

4.2.3.1 Children can do more if given the opportunity

A typical paper in this group is one by Carpenter, Ansell, Franke, Fennema, and Weisbeck (1993), which reported kindergarten children's problem-solving processes. The results suggest that "children can solve a wide range of problems, including problems involving multiplication and division situations, much earlier than generally has been presumed" (p. 439) and that "if specific multiplication and division schemata are required, these schemata are sufficiently well developed in many kindergarten children that they can solve multiplication and division problems by representing the action and relationships in the problems" (p. 440). The focus was on the problem-solving processes, but with an interest on "the potential for instruction to build upon and extend young children's problem solving processes" (p. 429).

Hativa and Cohen (1995) examined the feasibility of teaching certain negative number concepts and procedures to students of a much younger age than is presently done in schools and concluded with a positive answer. Working with low- and high-achievers, the study found that low-achievers gained at least as much as the high-achievers and suggested that teaching approaches based on students' pre-instructional intuitions can help students progress further than traditionally expected.

In a similar vein, Urbanska (1993) investigated the numerical competence of six-year-old children, concluding both that they have "a considerable degree of numerical competence" (p. 265) but that teachers in her study did not draw on this intuitive knowledge. Mulligan and Mitchelmore (1997, p. 328) argue, "the standard curriculum takes no advantage of the informal understanding that many students have developed well before grade 3."

Papers like these do not depart radically from the dominant theoretical views of the time. Rather, they expanded the boundaries of what could be said within those frameworks. This was a crucial contribution, enabling the idea that *children can do more* to be more readily accepted by the mathematics education community.

More directly to the point of algebra education, Mason (1991, 1996) reflects the optimistic point of view shared by most of the researchers in this group, that students come to school with natural powers of generalisation and abilities to express generality, and that the development of algebraic reasoning is, in large part, a matter of tapping into those naturally occurring capacities for didactic purposes. The pioneering work of Mason and his colleagues (Mason, 1989, 1991, 1996; Mason, Graham, Gower, & Pimm, 1985) provides a wide range of tasks and task-design principles that operationalise this fundamental observation. However, only more recently (with the notable exception of Davydov, discussed below), have there been empirical studies which explore learning and teaching implementing this approach. Much of this work has taken place in the USA and reflects the initiative of the National Council of Teachers of Mathematics (NCTM) in treating algebraic reasoning in a deliberately longitudinal way with roots in early mathematical development (NCTM 1989, 2000). This initiative is a response to a growing realisation of the failure of the approach to algebra in the USA, where it is introduced late, abruptly, and in relative isolation from other mathematics, with a focus on syntactic operation skill (see Kaput, 1998, 1999; Lacampagne, Blair, & Kaput, 1995; Moses, 1995; NCTM & Mathematical Sciences Education Board, National Research Council, 1998). Below, we will report further on the work in the USA spawned by this initiative and comparable efforts in other countries.

If those changes were mainly driven by curricula and failure considerations, work previously begun in the Soviet Union had been driven by theoretical considerations. Davydov's work precedes the more recent push towards building algebraic reasoning in elementary grades. A translation into English of *An experiment in introducing elements of algebra in elementary school* was published in 1962 in *Soviet Education* (Davydov, 1962) and in 1974 Hans Freudenthal published a paper on the Soviet work on the teaching of algebra at the lower grades of the elementary school (Freudenthal, 1974). However, it was only much later, in the 1990s that this work became better known in the West, and for this reason it will be considered among the happy stories.

The so-called Soviet School (Vygotsky, Luria, and Leontiev) proposed that learning precedes development, an assumption diametrically opposed to the

Piagetian idea that development precedes learning. Davydov and his colleagues aimed at producing an approach to primary mathematics that fostered generality in students' thinking by offering them thinking tools (socially, culturally, and historically developed) that could enable them to do things in mathematics that otherwise would take much longer. The case of the whole-part diagram and the use of literal notation are well known (Davydov, 1962). The title of his article, in English, is *An experiment in introducing elements of algebra in elementary school*, and its aim was to examine the level of generality those elements could bring to children's thinking. Davydov was quite naturally interested in primary school children, given his theoretical assumptions. So was Dienes (1973); the difference was that, while Davydov was centrally interested in fostering a mode of thinking, Dienes was interested in developing a concrete meaning for the rules of algebra, in a sense aiming primarily at content. Although Davydov's work showed that children could do more, unlike the papers mentioned above it actually bases that assumption on theoretical grounds. The key point is that children will, indeed, do more, if we offer them access to appropriate cultural tools—for instance, diagrams, and special notations.

This is the kind of research and development work that, so to speak, raised the banner *children can do more*, paving the way for early algebra as presented in this chapter. This was no small deal, given the dominant views at that time. While it is true today that most research in algebra education still falls outside early algebra (that is, it is still directed towards the education of older students), the work of these pioneering researchers and others opened the path for studying an early introduction to the ideas of algebra for mainstream students rather than merely for gifted students.

4.2.3.2 Opening the algebra education door even wider

As we said earlier, much of the research on algebra education up to the 1990s was dedicated either to producing systems of developmental stages or to producing catalogues of errors made by children. This work was oriented to the content of algebra and closely tied to the traditions of mathematics education, including historic relations between school arithmetic and school algebra.

For reasons not so easy to pin down, beginning in the late 1980s and continuing during the 1990s, this began to change, through the use of the history of mathematics as a source of insights into the difficulties students have with algebra (e.g., Gallardo, 1990; Radford, 1995; Sfard, 1995); through a more explicit discussion of the underlying epistemological aspects (e.g., Balacheff & Sutherland, 1994; Brousseau, 1983; Kaput, 1979; Lins, 1992, 1994, 2001; Radford, 1994; Sfard, 1991); and with the broadening of the discussion of linguistic aspects, beyond the usual syntax-semantics dichotomy (e.g., Arzarello, Bazzini, & Chiappini, 2001; Boero, 2001; Kirshner, 1990, 2001; Nemirovsky, 1996; Pimm, 1987; Chapter 9 in this book). Technology acted as a destabilising factor as well, and is discussed further below (see also Chapter 6 and Chapter 7 in this book).

Many of these issues emerged over a few years, in the sessions of the Algebra Working Group at conferences of the International Study Group on the Psychology of Mathematics Education (PME), supported by the diversity of backgrounds and interests of the researchers who took part in it. The book by Sutherland, Rojano, Bell, and Lins (2001) emerged from these discussions. This was most likely a real reflection of what was happening across the world in the algebra education research community.

This enriched perspective on algebra education allowed for more flexible views of what algebra education could or should mean. Levels of intellectual development were, more and more, being considered together with the effects of contexts. The use and learning of natural language was informing our understanding of the use of the language of algebra. We were getting insights on possible sources of difficulties from history. In addition, we observed that young students were facing difficulties that were previously faced by grown-up mathematicians. Had the human brain developed so quickly?

All this contributed to stimulate part of the mathematics education community to take, as we said, a more flexible view of algebra education, helping to open a door to early algebra.

4.2.3.3 New technologies: More challenges, more opportunities

From the late 1980s onwards, the increasing availability of computers and other technologies made it more and more appealing to consider algebra without necessarily associating it with the tradition of having manipulation of algebraic expressions as the core of algebra education. Certainly there was already a push in this direction (Fey, 1989; see also the review in Kaput, 1992), but the new technologies allowed students and teachers to integrate algebraic expressions into richer, more concrete and meaningful contexts, with much greater ease (Fey, 1984).

Probably the first widespread approach linking computers and algebra education was the use of programming languages (Camp & Machionini, 1984; Feurzieg, Lucas, Grant, & Faflick, 1969; Soloway, Lochhead, & Clement, 1982; Sutherland, 1989, 1993). This was quickly followed by the use of spreadsheets (Dettori, Garuti, & Lemut, 2001; Sutherland & Rojano, 1993) and specially designed software (Kieran, Boileau, & Garançon, 1996), as well as Computer Algebra Systems, some of which became embedded in hand-held devices, and used mainly with older students. Modelling and real data activities were also greatly stimulated (Kaput, 1994; Nemirovsky, 1996). A more extensive review here is unnecessary as the reader may consult Chapter 6 and Chapter 7 in this book that are specifically about these themes.

For the purposes of this chapter, what is important to emphasise is that the new technologies allow students to work with algebra in a variety of contexts, before they have mastered the by-hand manipulation of expressions. In addition, they greatly facilitate multi-representational activities (Kaput, 1986; also see the review in Kaput, 1992). The twofold implication is that, on the one hand, younger children can do these kinds of actions with the assistance of tools and, on the other hand, by doing them they will most likely develop an integrated perception of algebra and its applications, something found more difficult in the "first algebra, then applications" tradition, or even in the use of concrete settings to facilitate the transition or to help bridging the gap. The overall effect was further to call into question what is possible and appropriate with younger children, as well as raising questions of the nature of mathematics (Kaput, Noss, & Hoyles, 2001).

4.2.3.4 Conditions leading to change in algebra education

This combination of changes in our perception of children's thinking, changes in the scope and basis of research on children's thinking, and changes in the availability and roles of technologies, proved to be a powerful stimulus for algebra education. In a number of countries as mentioned above, the drive for reform of mathematics education in general, including mathematics teaching, has also provided adequate background for early algebra to grow, but we think that the underlying support came from the changes in foundational conditions mentioned above.

We suggest that a thorough study of this *transition* period could further enlighten our understanding of mathematics education today and towards the future. Given that algebra plays such a central role in school mathematics and in the thinking and planning of policy makers, curricula makers, teachers and pupils, and that it is associated, rightly or wrongly, with much of the failure in schools, there is a reasonable chance that such a study would be of interest to the community at large.

4.3 What Early Algebra Can Mean Today and For the Future

In this section, we will look at work representing that of people who participated in the Working Group on Early Algebra at the ICMI Study, reflecting, by and large, a shared perspective of *early algebra* as proposed in this chapter and amounting to an overview of the current early algebra landscape.

Research in and implementation of such a different perspective involves several considerations: for instance, integrating new instructional materials and/or teaching practices with existing ones, and perhaps replacing certain existing approaches with new ones. These considerations typically have multiple levels, ranging from detailed cognitive and classroom practice issues to larger scale questions associated with teacher professional development, assessment, and other systemic factors. Given the strong shift of perspective involved the discussion of early algebra naturally invites reflection at a theoretical level, an invitation that we accept.

A common assumption of the supporters of early algebra is that an *algebrafied* elementary mathematics would empower students, particularly by fostering a greater degree of generality in their thinking and an increased ability to communicate that generality. There are, however, two different views on what this algebrafication means and how it should proceed. One view is that we should build on what is already algebraic in young children's thinking, particularly with respect to their numerical, or arithmetical, reasoning. Another view is that changes in students' thinking are better promoted if we offer them tools such as notations and diagrams, which allow them to operate at a higher level of generality. In practice those views are not necessarily conflicting (Confrey, 1991).

Making this distinction highlights the need to examine the assumptions behind each of the views, leading to a clearer understanding of the possible roads for early algebra. Les Steffe (2001, p. 557) says, quite rightly, that school mathematics should be viewed "as a product of the functioning of children's intelligence". Steffe arrives at his statement from a Piagetian angle, and since there are different views on the functioning of children's intelligence, we are bound to find different specifications for early algebra. The suggestion remains, however, that a shift from content-centred planning would be of interest.

4.3.1 Arithmetic as a basis for early algebra

A group of papers focus on what is algebraic in arithmetic, that is, what can we find in arithmetic that may serve as the basis for developing students' algebraic understanding. Fujii (2003) and Fujii and Stephens (2001) propose the notion of *quasi-variables*, which are numbers within "a number sentence or group of number sentences that indicate an underlying mathematical relationship which remains true whatever the numbers used are" (p. 259). An example of such sentences is:

$$78 - 49 + 49 = 78$$

where both 78 and 49 can be considered as acting as quasi-variables, indicating the relationship that a number (e.g., 78) remains unchanged if something (e.g., 49) is subtracted and then added to it. They observe that the intention is not to introduce children to expressions like $a - b + b = a$, but rather to get them to understand that this sentence belongs to a *type* of number sentence which is true whatever number is taken away and added back. Carpenter and Levi (1999) provided a comparable analysis. More general task-design principles suggesting sequences of unexecuted number sentences have been offered by Blanton and Kaput (in press) and Thompson (1993). The key idea is that children can become acquainted with the important concept of variable either well before they are introduced to formal algebraic notation or as an intrinsic part of learning variation.

Brizuela and Lara-Roth (2001a, 2001b), as part of a larger team including Carraher, Schliemann, and others operating from the same general principle (see Carraher, Brizuela, & Earnest, 2001; Carraher, Schliemann, & Brizuela, in press), explicitly state their interest in bringing out the algebraic character of arithmetic. Working with additive relations and function tables, Brizuela and Lara-Roth's focus is on "uncovering the understandings already present by analysing the original self-designed tables constructed by young children" (2001a, p. 111). Similarly, Carraher, Brizuela, and Earnest (2001) worked with young children on the notion of difference, developing in the process what they termed *variable number lines*, number lines in which instead of specific numbers there are expressions like N – 3, N – 2, N – 1, N, and N + 1. Schliemann, Lara-Roth, and Goodrow (2001) explored multiplication tables as function tables. Marjanovic and Kadijevich (2001) offer five topics through which arithmetic can be linked to algebra: first steps in addition and subtraction, the invariant manner of expressing arithmetic procedures explicitly (an approach similar to Fujii and Stephens' quasi-variables), equations, inequalities, and discovering a rule.

Although also working in a numerical context, Carpenter and Franke (2001) focus on the processes of generalisation and proof, addressing the aspect of algebra that is generalised arithmetic: "We characterise the development of elementary school children's algebraic reasoning as reflected in their ability to generate and justify generalisations about fundamental properties of arithmetic" (p. 155). Warren (2001) examines children's understanding of the commutative law in the early years and just before they begin formal algebra studies, pointing to several implications and recommendations for teaching and learning algebra.

4.3.2 Algebrafying the elementary mathematics experience

Those papers share the idea that the study of arithmetic, both numbers and operations, already involves a degree of generalisation and thus a useful way into algebra is to exploit that generality by building and expressing new generalisations. This is a central idea in what Kaput and Blanton (2001) call "algebrafying the elementary mathematics experience" (p. 344). They propose that this process has three dimensions: (1) The process of building task-opportunity for generalisation and progressive formalisation of mathematical patterns and structure; (2) Building teachers' *algebra eyes and ears* so that they can recognise opportunities for such work in daily practice; and (3) Creating classroom practice and culture that support such work. They also argue that introducing algebra early would open curricular space needed at the secondary level and add a new level of coherence, depth, and power to elementary mathematics. Crucially, it is necessary for "democratising access to powerful ideas [...] thereby making opportunities for achievement more equitable" (p. 345).

Blanton and Kaput (2001) describe the implementation of early algebra by a teacher with her grade 3 students (8-9 years old) prior to its implementation on a district-wide scale, a task that involves deep changes in the practice and thinking of teachers. It also illustrates the point that the changes needed to implement early algebra involve many systemic educational factors beyond classroom teaching, curriculum, and learning. It should be noted, however, that many of these factors vary greatly from country to country (see Chapter 13 in this book).

Lee (2001), in discussing six views of what algebra means, proposes that *Algebra as a culture* makes it possible to pull together the other five views (as a language, as a way of thinking, as a kind of activity, as a tool, and as generalised arithmetic), "and weave them into a rich tapestry of what early algebra is or might become" (p. 397). From there, she tells an algebra story (first proposed by Kaput and Blanton (2001)) for elementary school, involving engagement in algebraic activities and communication in an algebraic language.

Taking *a culture of algebra* from a slightly different point of view, Lins (2001) proposes the notion of *legitimacy* as crucial in algebra education, arguing that an early introduction to the culture of algebra promotes, for instance, a natural legitimacy for calculating with letters. He also argues that students' difficulties documented by research may have a strong root in the fact that teachers too often fail to make clear to students the subtle shifts in the mode of meaning production. An example is shifting between *equation as scale balance* and *equation as a numerical sentence*. This suggestion is consistent with the finding that children can do more if given the opportunity and supports the value of viewing early algebra as a process of enculturation, allowing for the integration of *algebrafying from arithmetic*—as found in most papers discussed in this section—with *algebraic objects as tools for general thinking in problem-solving*—as found in the work of Davydov.

4.4 A Research Agenda

In view of what we have considered so far in this chapter, three broad areas seem to deserve attention from the algebra education community in the coming years.

The first is assessment and curriculum development from the point of view of *early algebra* seen as an *early start in algebra education*. The second is the study of the relationship among research, policies, and practice in this context, with special attention to teacher education, both for beginning and for experienced teachers. The third is a study of the implications for later grades of changes in earlier grades. This would naturally involve again the two areas identified above, now with respect to those older students. And of course, if we believe that younger students can do more then, perhaps, we have set the stage for also believing that older students can do more. Indeed, it should be recalled that most of the sad stories of Section 4.2 concerned older students, products of the current algebra education system, whereas most of the happy stories concerned younger students whose mathematical experiences vary significantly from the traditional norm.

In the sub-sections that follow, we do not mean to be exhaustive in any sense. Also, the specific suggestions presented are meant only as examples, to help to clarify the scope of those specific sub-areas. Research related to environments incorporating new technologies and to teacher education is quite relevant to all sub-sections but for these we refer the reader to Chapters 6, 7, and 10 in this volume.

4.4.1 Basic yet practical research

Basic yet practical research is needed on cognition, development, culture and change in mathematics education. We suggest that algebra education may be a venue for fundamental research while at the same time being of highly practical value. Algebra has historically been at the centre of the debates about what children can and cannot do at given ages or levels of development, and as we have seen, because of its use of powerful cultural tools, it raises deep issues regarding the relation between cultural tools and development. Analysis of these issues might lead to better theoretical syntheses than have been achieved to date. Furthermore, given the centrality of algebraic reasoning to mathematics itself and hence to school mathematics, understanding how it develops will serve mathematics education more broadly. Lastly, studying the many issues that arise when such fundamental change as the introduction of early algebra is attempted, can lead to much understanding with practical value, particularly if international differences are kept in mind so that these issues are understood in full generality and more robustly.

4.4.2 Research on forms of algebraic thinking

Another aspect of research should focus on the *teaching* side, in an effort to anticipate those aspects of algebraic thinking that could or should be presented, promoted and emphasised in the classroom. It would draw both from research indicated in Section 4.4.1 and other studies.

Several directions can be pursued. For instance, one might be interested in integration between algebra and other content areas, not only arithmetic, but geometry or the mathematics of data, for instance. Or one might be interested in which tools (diagrams, notations, graphs) can successfully lead students to develop more powerful, general algebraic ways of thinking. Or one might be interested in the enculturation aspect such as getting students to be familiar with the use of literal notation in different contexts, beginning working with *quasi-variables* for example, and getting students familiar with the idea of reasoning from the *forms* of expressions, and directly manipulating expressions (including numerical ones) to obtain new, and hopefully more useful, ones.

Of course, those same aspects could also be and have been of interest for someone focusing on older students' algebra education. The fact that we are here talking about much younger students, however, suggests that this area of research should be considered afresh.

4.4.3 Students failing in algebra

As we have pointed out in Section 4.2, much research has been conducted in the past concerning students' error patterns and misconceptions and concerning stages or levels of development. This effort seemed directed towards improving algebra education by anticipating the bad things that could happen in the classrooms and recognising putative developmental constraints on students' learning.

Few questions, however, seem to have been posed to the students themselves, especially those who are failing, regarding what sense they make of their condition. Research on students' beliefs has usually been directed to general beliefs about mathematics, not algebra (for a recent and similar example from science education, see Davis (2003)). We can ask, for instance, in their terms "What factors do they attribute their failure to? What things in algebra do and do not make sense to them and why?" Much more could be investigated here, including eliciting how students categorise the things that are being presented to them. It is difficult to anticipate what kind of insights we will get from research in this area, but we should at least expect to develop better ways of *reading* students' thinking. This may make it possible, perhaps, to promote *non-deficit reading* as an idea that is useful in the classrooms. This means looking at what students are *actually* thinking about and with, rather than at what they are failing to do and checking this against what they are *expected* to do. Franke and Carey (1997), while examining young children's perception of mathematics in problem-solving environments, provide an interesting example of the kind of research meant in this sub-section.

4.4.4 Curriculum development and intervention studies

In Section 4.4.1 we pointed to theoretical development, in Section 4.4.2 we pointed to the teachers' side, and in Section 4.4.3 we pointed to the students' side. In Section 4.4.4 these three areas come together to inform curriculum development based on long-term intervention studies. Different trails followed in the three previous sub-sections will probably lead to different approaches to this theme.

One question could be how to algebrafy the whole (or parts) of early mathematics. Another one might be how to combine different traditions to produce innovative and efficient approaches such as implementing the Davydov-Elkonin curriculum in a way compatible with Western traditions in school mathematics. A major effort along these lines, the Measure Up Project, is underway at the University of Hawaii led by Dougherty and colleagues (Dougherty & Zilliox, 2003; see also Chapter 5). Questions like "Does traditional arithmetic affect some children's abilities to reason algebraically?" and "How do mathematical learning and development evolve in early algebra environments?" are also good pointers to the kind of studies we are suggesting in this sub-section. Carraher, Schliemann, Brizuela, and colleagues are addressing these kinds of questions in a longitudinal

project studying student's progress through the elementary grades with a consistent set of new curriculum materials (Carraher, Schliemann, & Brizuela, in press).

It seems clear, however, that a substantial part of this effort should explicitly take into consideration how strongly traditional views are present in schools, and that includes a permanent concern with the interface between the *new* and the *old*. Studies that look at how teachers adapt to change in early mathematics (towards early algebra) are particularly relevant. Three large scale efforts along these lines are currently underway, one by Blanton and Kaput (2002, in press), another by Carpenter and Franke (2001), and a third by Schifter, Bastable, and Monk that is extending the *Developing Mathematical Ideas* teacher professional development program to early algebra (see Schifter, 1999, for an illustration of the use of teachers' own writings and voices in the style of this effort).

4.4.5 Implications of early algebra for later grades

Changes in curricula for earlier grades naturally have implications for later grades. In this case, there are good reasons to believe that early algebra would have a significant impact on the curricula for those later grades, for two reasons. First, topics that the students would traditionally meet at later grades already will have been studied. That does not mean those topics will have been explored fully, rather that at later grades there may be further studies of those topics instead of an introduction to them. This may require major changes in curricula. Second, it is reasonable to suppose that the exposure of young students to algebra, even if only to some aspects of it, is bound to change their thinking about other topic areas in many ways. For instance, based on ongoing work at many sites, it is likely that their numerical thinking will generally be different from that of students who have not been exposed to early algebra, as will their thinking about such transition problem areas such as the equal sign. Finally, of course, there is the issue of what new kinds of ideas will be within the capacity of students whose introduction to algebraic thinking began in the early grades. Advocates of and researchers in early algebra should pay close attention to such issues, as these will certainly be part of the process of trying to implement the proposed changes for the long term.

4.4.6 Policy and practice in the context of change

Finally, there is scope for studies that tackle the complex relationship among all the issues presented in Sections 4.4.1 to 4.4.5, including the implementation processes, in which the matter of policy is central. Such complex studies are usually best conducted by larger groups working in collaboration, possibly involving multiple countries for the reasons mentioned earlier. Indeed, plans for such collaboration are underway as of this writing. This is especially important since the potential impact of the changes implied by early algebra could encompass the whole of mathematics education in schools. Such studies will benefit from what is already known from previous major curricular reforms and will inform future efforts. In a sense, that closes the circle: the issues in Section 4.4.1 can be said to be more theoretical, while in this sub-section we face the systemic and institutional issues which, while of a much more practical nature, are themselves subject to substantial theory development.

4.5 Final Remarks

It is fair to say that the proposition of an early start to algebra has various roots, and it is safe to say that its branches have broad and far-reaching implications. On the one hand, roots will be found in theoretical developments that led to curricular development, as in the case of the so-called Soviet School. On the other hand, roots will also be found in studies (markedly the studies in the spirit of the happy stories of Section 4.2.3) that led to the reconsideration of some theoretical assumptions. We reiterate our suggestion that algebra education may be an ideal place for the interplay of theory and practice.

In both cases there is a common consequence: the strengthening of the idea that young children can do more than we expected before. That, in itself, can answer the question "Why early algebra?" simply because our students deserve the chance to develop to the best of their potential.

Besides everything said so far in this chapter, we emerge from this process with a renewed awareness of the need to pay attention to what our students *are being*, rather than focusing on anticipating what they are, are not, or will be. That makes the suggested research of Section 4.4.3, which proposes research into students' perspectives, a rather intriguing area to be investigated.

Finally, if it is not yet sufficiently clear, early algebra as proposed here aims at promoting flexible, articulated, and powerful thinking (with emphasis on generality, a central aspect of what makes mathematical thinking), not at *making kids better in algebra manipulation*. Technique is only a part of the story and is certainly not the main target.

4.6 References

Arzarello, F., Bazzini, L., & Chiappini, G. (2001). A model for analysing algebraic processes of thinking. In R. Sutherland, T. Rojano, A. Bell, & R. Lins (Eds.), *Perspectives on school algebra* (pp. 61-82). Dordrecht, The Netherlands: Kluwer Academic.

Balacheff, N. (2001). Symbolic arithmetic vs algebra: The core of a didactical dilemma. In R. Sutherland, T. Rojano, A. Bell, & R. Lins (Eds.), *Perspectives on school algebra* (pp. 249-260). Dordrecht, The Netherlands: Kluwer Academic.

Balacheff. N., & Sutherland, R. (1994). Epistemological domain of validity of Microworlds: The case of LOGO and Cabri-géometre. In R. Lewis & P. Mendelson (Eds.), *Lessons from learning* (pp. 137-150). North Holland: Elsevier Science B.V.

Becker, G. (1988). A classification of students' errors in secondary level algebra. In A. Borbás (Ed.), *Proceedings of the 12th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 131-138). Veszprém, Hungary: Program Committee.

Biggs, J., & Collis, K. (1982). *Evaluating the quality of learning: The SOLO Taxonomy*. New York: Academic Press.

Blanton, M., & Kaput, J. (2001). Algebrafying the elementary mathematics experience. Part II: Transforming practice on a district-wide scale. In H. Chick, K. Stacey, J. Vincent, & J. Vincent (Eds.), *The future of the teaching and learning of algebra* (Proceedings of the 12th ICMI Study Conference, pp. 87-95). Melbourne, Australia: The University of Melbourne.

Blanton, M., & Kaput, J. (2002). Design principles for tasks that support algebraic thinking in elementary school classrooms. In A. D. Cockburn & E. Nardi (Eds.), *Proceedings of the 26th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, 105–112). University of East Anglia: Norwich, UK.

Blanton, M., & Kaput, J. (in press). Design principles for instructional contexts that support students' transition from arithmetic to algebraic reasoning. Elements of task and culture. In R. Nemirovsky, B. Warren, A. Rosebery, & J. Soloman (Eds.), *Learning environments: The encounter of everyday and disciplinary experiences*. Mahwah, NJ: Lawrence Erlbaum.

Boero, P. (2001). Transformation and anticipation as key processes in algebraic problem solving. In R. Sutherland, T. Rojano, A. Bell, & R. Lins (Eds.), *Perspectives on school algebra* (pp. 99-120). Dordrecht, The Netherlands: Kluwer Academic.

Booth, L. (1984). *Algebra: Children's strategies and errors*. Windsor, UK: NFER-NELSON.

Booth, L. (1989). Grade 8 students' understanding of structural properties in mathematics. In G. Vergnaud, J. Rogalski, & M. Artigue (Eds.), *Proceedings of the 13th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 141-148). Paris, France: Program Committee.

Brizuela, B., & Lara-Roth, S. (2001a). Additive relations and function tables. In H. Chick, K. Stacey, J. Vincent, & J. Vincent (Eds.), *The future of the teaching and learning of algebra* (Proceedings of the 12th ICMI Study Conference, pp. 110-119). Melbourne, Australia: The University of Melbourne.

Brizuela, B. M., & Lara-Roth, S. (2001b). Additive relations and function tables. *Journal of Mathematical Behavior, 20*(3), 309-319.

Brousseau, G. (1983). Les Obstacles epistemologiques et les problemes en mathematiques. *Revista Reserches en Didactique des Mathématiques, 4*(2), 165-180.

Camp, J., & Marchionini, G. (1984). Programming and learning: Implications for mathematics education. In V. P. Hanson & M. J. Zweng (Eds.), *Computers in mathematics education, 1984 Yearbook* (pp. 118-126). Reston, VA: National Council of Teachers of Mathematics.

Carpenter, T., Ansell, E., Franke, M., Fennema, E., & Weisbeck, L. (1993). Models of problem-solving: A study of kindergarten children's problem-solving processes. *Journal for Research in Mathematics Education, 24*, 429-441.

Carpenter, T., & Franke, M. (2001). Developing algebraic reasoning in the elementary school: Generalization and proof. In H. Chick, K. Stacey, J. Vincent, & J. Vincent (Eds.), *The future of the teaching and learning of algebra* (Proceedings of the 12th ICMI Study Conference, pp. 155-162). Melbourne, Australia: The University of Melbourne.

Carpenter, T. P., & Levi, L. (1999). *Developing conceptions of algebraic reasoning in the primary grades*. Paper presented at the Annual Meeting of the American Educational Research Association, Montreal, Canada (Available from the first author at the University of Wisconsin-Madison).

Carraher, D., Brizuela, B., & Earnest, D. (2001). The reification of additive differences in early algebra: Viva La Différence! In H. Chick, K. Stacey, J. Vincent, & J. Vincent (Eds.), *The future* *of the teaching and learning of algebra* (Proceedings of the 12th ICMI Study Conference, pp. 163-170). Melbourne, Australia: The University of Melbourne.

Carraher, D., Schliemann, A.D., & Brizuela, B.M. (in press). Treating operations as functions. In D. W. Carraher, R. Nemirovsky, & C. DiMattia, C. (Eds.), *Media and Meaning*. CD-ROM issue of Monographs for the Journal of Research in Mathematics Education.

Confrey, J. (1991). Steering a course between Vygotsky and Piaget. *Educational Researcher, 20*(8), 28-32.

Davis, E. (2003). Untangling dimensions of middle school students' beliefs about scientific knowledge and science learning. *International Journal for Science Education, 25*(4), 439-468.

Davis, R. B. (1975). Cognitive processes involved in solving simple algebraic equations. *Journal of Children's Mathematical Behavior, 1*(3), 7-35.

Davis, R. B. (1984). ICME-5 report: Algebraic thinking in early grades. *The Journal of Mathematical Behavior, 4*, 195-208.

Davydov, V. (1962). An experiment in introducing elements of algebra in elementary school. *Soviet Education, 8*, 27-37.

Davydov, V. (1975). The psychological characteristics of the "prenumerical" period of mathematics instruction. In L. P. Steffe (Ed.), *Soviet studies in the psychology of learning and teaching mathematics* (Vol. 7, pp. 109-206). Chicago, IL: The University of Chicago Press.

Davydov, V. (1982). The psychological characteristics of the formation of elementary mathematical operations in children. In T. Carpenter, J. Noser, & T. Romberg (Eds.), *Addition and subtraction: A cognitive perspective* (pp. 224-238). Hillsdale, NJ: Lawrence Erlbaum Associates.

Davydov, V. (1983). A concept of educational activity for school children. *Soviet Psychology, 11*(2), 50-76.

Dettori, G., Garuti, R., & Lemut, E. (2001). From arithmetic to algebraic thinking by using a spreadsheet. In R. Sutherland, T. Rojano, A. Bell, & R. Lins (Eds.), *Perspectives on school algebra* (pp. 191-208). Dordrecht, The Netherlands: Kluwer Academic.

Dienes, Z. (1973). *The six stages in the process of learning mathematics* (P. L. Seaborne, Trans.). New York: Humanities Press.

Dougherty, B. J., & Zilliox, J. (2003). Voyaging from theory to practice in teaching and learning: A view from Hawai'i. In N. Pateman, B. Dougherty, & J. Zilliox (Eds.), *Proceedings of the 27th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, 13-31). Honolulu, Hawaii: Program Committee.

Feurzeig, W., Lucas, G., Grant, R., & Faflick, P. (1969). *Programming languages as a conceptual framework for teaching mathematics.* (Technical Report No. 1899). Cambridge, MA: Bolt, Beranek, & Neuman.

Fey, J. T. (1984). *Computing and mathematics: The impact on secondary school curricula.* Reston, VA: National Council of Teachers of Mathematics.

Fey, J. (1989). Technology and mathematics education: A survey of recent developments and important problems. *Educational Studies in Mathematics, 20*, 237-272.

Filloy, Y. (1987). Modelling and the teaching of algebra. In J. Bergeron, N. Herscovics, & C. Kieran (Eds.), *Proceedings of the 11th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. I, pp. 295-300). Montreal, Canada: Program Committee.

Franke, M., & Carey, D. (1997). Young children's perceptions of mathematics in problem-solving environments. *Journal for Research in Mathematics Education, 28*(1), 8-25.

Freudenthal, H. (1974). Soviet research on teaching algebra at the lower grades of elementary school. *Educational Studies in Mathematics, 5*, 391-412.

Fujii, T. (2003) Probing students' understanding of variables through cognitive conflict problems: Is the concept of a variable so difficult for students to understand? In N. Pateman, B. Dougherty, & J. Zilliox (Eds.), *Proceedings of the 27th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 49-66). Honolulu, Hawaii: Program Committee.

Fujii, T., & Stephens, M. (2001). Fostering an understanding of algebraic generalisation through numerical expressions: The role of quasi-variables. In H. Chick, K. Stacey, J. Vincent, & J. Vincent (Eds.), *The future of the teaching and learning of algebra* (Proceedings of the 12th ICMI Study Conference, pp. 258-264). Melbourne, Australia: The University of Melbourne.

Gallardo, A. (1990). Avoidance and acknowledgement of negative numbers in the context of linear equations. In G. Booker, P. Cobb, & T. de Mendicuti (Eds.), *Proceedings of the 14th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 43-50). Oaxtepex, Mexico: Program Committee.

Gallardo, A., & Rojano, T. (1987). Common difficulties in the learning of algebra among children displaying low and medium pre-algebraic proficiency levels. In J. Bergeron, N. Herscovics, & C. Kieran (Eds.), *Proceedings of the 11th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. I, pp. 301-307). Montreal, Canada: Program Committee.

Garcia, R., & Piaget, J. (1984). *Psicogénesis e História de la Ciencia*. Mexico City, Mexico: Siglo Veintuno Editores.

Harper, E. (1987). Ghosts of Diophantus. *Educational Studies in Mathematics, 18*, 75-90.

Hart, K. (1984). *Children's understanding of mathematics: 11-16*. London: John Murray..

Hativa, N., & Cohen, D. (1995). Self learning of negative number concepts by lower division elementary students through solving computer-provided numerical problems. *Educational Studies in Mathematics, 28*, 401-431.

Heath, T. (1964). *Diophantus of Alexandria: A study in the history of Greek algebra.* New York: Dover.

Herscovics, N. (1989). Cognitive obstacles encountered in the learning of algebra. In C. Kieran & S. Wagner (Eds.), *Research issues in the learning and teaching of algebra* (pp. 60-86). Reston, VA: National Council of Teachers of Mathematics, Hillsdale, NJ: Lawrence Erlbaum Associates.

Kaput, J. (1979). Mathematics and learning: Roots of epistemological status. In J. Clement & J. Lochhead (Eds.), *Cognitive process instruction* (pp. 289-303). Philadelphia, PA: Franklin Institute Press.

Kaput, J. (1986). Information technology and mathematics: Opening new representational windows. *The Journal of Mathematical Behavior, 5*(2), 187–207.

Kaput, J. (1992). Technology and mathematics education. In D. Grouws (Ed.), *Handbook on research in mathematics teaching and learning* (pp. 515-556). New York: Macmillan.

Kaput, J. (1994). The representational roles of technology in connecting mathematics with authentic experience. In R. Biehler, R. W. Scholz, R. Strasser, & B. Winkelman (Eds.), *Mathematics didactics as a scientific discipline* (pp. 379-397). Dordrecht, The Netherlands: Kluwer Academic.

Kaput, J. (1998). Transforming algebra from an engine of inequity to an engine of mathematical power by "algebrafying" the K–12 curriculum. In National Council of Teachers of Mathematics and Mathematical Sciences Education Board (Eds.), *The nature and role of algebra in the K–14 curriculum: Proceedings of a national symposium* (pp. 25-26). Washington, DC: National Research Council, National Academy Press.

Kaput, J. (1999). Teaching and learning a new algebra. In E. Fennema & T. Romberg (Eds.), *Mathematics classrooms that promote understanding* (pp. 133-155). Mahwah, NJ: Lawrence Erlbaum Associates.

Kaput, J., & Blanton M. (2001). Algebrafying the elementary mathematics experience. Part I: Transforming task structures. In H. Chick, K. Stacey, J. Vincent, & J. Vincent (Eds.), *The future of the teaching and learning of algebra* (Proceedings of the 12th ICMI Study Conference, pp. 344-351). Melbourne, Australia: The University of Melbourne.

Kaput, J., Noss, R., & Hoyles, C. (2001). Developing new notations for learnable mathematics in the computational era. In L. D. English (Ed.), *The handbook of international research in mathematics* (pp. 51-73). London: Kluwer Academic.

Kieran, C. (1981). Pre-algebraic notions among 12- and 13-year-olds. In Equipe de Recherche Pedagogique Laboratoire (Eds.), *Proceedings of the 5th annual conference of the International Group for the Psychology of Mathematics Education* (pp. 158-164). Grenoble, France: Program Committee.

Kieran, C., Boileau, A., & Garançon M. (1996). Introducing algebra by means of a technology-supported functional approach. In N. Bednarz, C. Kieran, & L. Lee (Eds.), *Approaches to algebra* (pp. 257-293). Dordrecht, The Netherlands: Kluwer Academic.

Kirshner, D. (1987). The myth about binary representation in algebra. In J. Bergeron, N. Herscovics, & C. Kieran (Eds.), *Proceedings of the 11th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. I, pp. 308-315). Montreal, Canada: Program Committee.

Kirshner, D. (1990). Acquisition of algebraic grammar. In G. Booker, P. Cobb, & T. de Mendicuti (Eds.), *Proceedings of the 14th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 75-84). Oaxtepex, Mexico: Program Committee.

Kirshner, D. (2001). The structural algebra option revisited. In R. Sutherland, T. Rojano, A. Bell, & R. Lins (Eds.), *Perspectives on school algebra* (pp. 83-98). Dordrecht, The Netherlands: Kluwer Academic.

Klein, J. (1968). *Greek mathematical thought and the origins of algebra.* Cambridge, MA: The MIT Press.

Küchemann, D. (1978). Children's understanding of numerical variables. *Mathematics in School,* September, 23-26.

Küchemann, D. (1984). Stages in understanding algebra. *Journal of Structured Learning, 8,* 113-124.

Lacampagne, C., Blair, W., & Kaput, J. (Eds.). (1995). *The algebra initiative colloquium: Vols. 1 and 2.* Washington, DC: U.S. Department of Education.

Lee, L. (1997). *Algebraic understanding: The search for a model in the mathematics education community.* Unpublished doctoral thesis. Université du Québec à Montréal, Canada.

Lee, L. (2001). Early algebra - But which algebra? In H. Chick, K. Stacey, J. Vincent, & J. Vincent (Eds.), *The future of the teaching and learning of algebra* (Proceedings of the 12th ICMI Study Conference, pp. 392-399). Melbourne, Australia: University of Melbourne.

Lewis, C. (1981). Skill in algebra. In J. R. Anderson (Ed.), *Cognitive skills and their acquisition* (pp. 85-110). Hillsdale, NJ: Lawrence Erlbaum Associates.

Lins, R. (1992.) *A framework for understanding what algebraic thinking is.* Unpublished Ph.D. Thesis. Shell Centre for Mathematical Education/Nottingham University, UK.

Lins, R. (1994). Eliciting the meanings for algebra produced by students: Knowledge, justification and semantic fields. In J. P. da Ponte, & J. F. Matos (Eds.), *Proceedings of the 18th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. III, pp. 184-191). Lisbon, Portugal: Program Committee.

Lins, R. (2001). The production of meaning for algebra: A perspective based on a theoretical model of semantic fields. In R. Sutherland, T. Rojano, A. Bell, & R. Lins (Eds.), *Perspectives on school algebra* (pp. 37-60). Dordrecht, The Netherlands: Kluwer.

Marjanovic, M., & Kadijevich, D. (2001). Linking arithmetic to algebra. In H. Chick, K. Stacey, J. Vincent, & J. Vincent (Eds.), *The future of the teaching and learning of algebra* (Proceedings of the 12th ICMI Study Conference, pp. 425-429). Melbourne, Australia: The University of Melbourne.

Mason, J. (1989). Mathematical abstraction as the result of a delicate shift of attention. *For the Learning of Mathematics, 9*(2), 2-8.

Mason, J. (1991). *Supporting primary mathematics: Algebra.* Milton Keynes, UK: Open University Press.

Mason, J. (1996). Expressing generality and roots of algebra. In N. Bednarz, C. Kieran, & L. Lee (Eds.), *Approaches to algebra: Perspectives for research and teaching* (pp. 65-86). Dordrecht, The Netherlands: Kluwer.

Mason, J., Graham, A., Gower, N., & Pimm, D. (1985). *Routes to / roots of algebra.* Milton Keynes, UK: Open University Press.

Matz, M. (1980). Towards a computational theory of algebraic competence. *Journal of Mathematical Behavior, 3,* 93-166.

Matz, M. (1982). Towards a process model for high school algebra errors. In D. H. Sleeman & J. S. Brown (Eds.), *Intelligent tutoring systems* (pp. 25-50). New York: Academic Press.

McArthur, D., Stasz, C., & Zmunidzinas, M. (1990). Tutoring techniques in algebra. *Cognition and Instruction, 7,* 197-244.

Moses, R. (1995). Algebra, the new civil right. In C. Lacampagne, W. Blair, & J. Kaput (Eds.), *The algebra colloquium: Vol. 2. Working group papers* (pp. 53-67). Washington, DC: U.S. Department of Education.

Mulligan, J., & Mitchelmore, M. (1997). Young children's intuitive models of multiplication and division. *Journal for Research in Mathematics Education, 28,* 309-330.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA: Author.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* Reston, VA: Author.

National Council of Teachers of Mathematics, & Mathematical Sciences Education Board, National Research Council (Eds.). (1998). *The nature and role of algebra in the K–14 curriculum: Proceedings of a national symposium.* Washington: National Academy Press.

Nemirovsky, R. (1996). A fundamental approach to algebra: Two issues that emerge. In N. Bednarz, C. Kieran, & L. Lee (Eds.), *Approaches to algebra* (pp. 295-313). Dordrecht, The Netherlands: Kluwer Academic.

Pereira-Mendoza, L. (1987). Error patterns and strategies in algebraic simplification. In J. Bergeron, N. Herscovics, & C. Kieran (Eds.), *Proceedings of the 11th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. I, pp. 331-337). Montreal, Canada: Program Committee.

Petitto, A. (1979). The role of formal and non-formal thinking in doing algebra. *Journal of Children's Mathematical Behavior, 2*(2), 69-82.

Pimm, D. (1987). *Speaking mathematically.* London: Hodder & Stoughton.

Radford, L. (1994). Moving through systems of mathematical knowledge: From algebra with one unknown to algebra with two unknowns. In J. P. da Ponte & J. F. Matos (Eds.), *Proceedings of the 18th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. IV, pp. 73-80). Lisbon, Portugal: Program Committee.

Radford, L. (1995). Before the other unknowns were invented: Didactic inquiries in the methods and problems of medieval Italian algebra. *For the Learning of Mathematics*, *15*(3), 28-38.

Schifter, D. (Ed.). (1999). *Voicing the new pedagogy: Teachers interpret the rhetoric of mathematics education reform*. New York: Teachers College Press.

Schliemann, A., Lara-Roth, S., & Goodrow, A. (2001). Tables as multiplicative function tables. In H. Chick, K. Stacey, J. Vincent, & J. Vincent (Eds.), *The future of the teaching and learning of algebra* (Proceedings of the 12th ICMI Study Conference, pp. 531-540). Melbourne, Australia: The University of Melbourne.

Sfard, A. (1989). Transition from operational to structural conception: The notion of function revisited. In G. Vergnaud, J. Rogalski, & M. Artigue (Eds.), *Proceedings of the 13th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 151-158). Paris, France: Program Committee.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, *22*, 1 - 36.

Sfard, A. (1995). The development of algebra: Confronting historical and psychological perspectives. In *New perspectives on school algebra: Papers and discussions of the ICME-7 algebra working group. Journal of Mathematical Behavior (Special Issue)*, *14*, 15-39.

Sleeman, D. (1986). Introductory algebra: A case study of misconceptions. *Journal of Mathematical Behavior*, *5*(1), 25-52.

Soloway, E., Lochhead, J., & Clement, J. (1982). Does computer programming enhance problem solving ability? Some positive evidence on algebra word problems. In R. J. Seidel, R. E. Anderson, & B. Hunter (Eds.), *Computer literacy* (pp. 171-185). New York: Academic Press.

Steffe, L. P. (2001). What is algebraic about children's numerical operating? In H. Chick, K. Stacey, J. Vincent, & J. Vincent (Eds.), *The future of the teaching and learning of algebra* (Proceedings of the 12th ICMI Study Conference, pp. 556-563). Melbourne, Australia: The University of Melbourne.

Sutherland, R. (1989). Providing a computer-based framework for algebraic thinking. *Educational Studies in Mathematics*, *20*, 317-344.

Sutherland, R. (1993). Connecting theory and practice: Results from the teaching of LOGO. *Educational Studies in Mathematics*, *24*, 95-113.

Sutherland, R., & Rojano, T. (1993). Spreadsheet approach to solving algebra problems. *The Journal of Mathematical Behavior*, *12*(3), 353-383.

Sutherland, R., Rojano, T., Bell, A., & Lins, R. (2001). *Perspectives on school algebra*. Dordrecht, The Netherlands: Kluwer Academic

Thompson, P. W. (1993). Quantitative reasoning, complexity, and additive structures. *Educational Studies in Mathematics, 25*(3), 165–208.

Urbanska, A. (1993). On the numerical competence of six-year-old children. *Educational Studies in Mathematics*, *24*, 265-275.

Wagner, S. (1981). Conservation of equation and function under transformations of variable. *Journal for Research in Mathematics Education, 12*, 107-118.

Warren, E. (2001). Algebraic understanding: The importance of learning in the early years. In H. Chick, K. Stacey, J. Vincent, & J. Vincent (Eds.), *The future of the teaching and learning of algebra* (Proceedings of the 12th ICMI Study Conference, pp. 633-640). Melbourne, Australia: The University of Melbourne.

Wenger, R. (1987). Some cognitive science perspectives concerning algebra and elementary functions. In A. Schoenfeld (Ed.), *Cognitive science and mathematics education* (pp. 217-252). Hillsdale, NJ: Lawrence Erlbaum Associates.