# CHARACTERISING THE MATHEMATICS OF THE MATHEMATICS TEACHER FROM THE POINT OF VIEW OF MEANING PRODUCTION

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## INTRODUCTION

The professional development of mathematics teachers—and by this I am mean both pre-service and in-service development—is currently an issue of major interest for the mathematics education community; one sign of such relevance is the existence of an international journal specifically dedicated to this field[2], as well as a forthcoming ICMI Study on The Professional Education and Development of Teachers of Mathematics.[3]

It is not difficult to understand why this should be a key area of mathematics education. As most of the mathematical education in different countries happens within school systems, that is, in a systematic way, it would seem necessary to provide specific preparation of and support to the professionals that will face the task of realising that education.[4]

Surprisingly—perhaps given the seemingly obvious character of that statement—, until not so long ago the bulk of published research in mathematics education did not deal directly with the issue of the professional development of mathematics teachers. Issues of cognitive development, errors/misconceptions and teaching strategies, for instance, largely dominated the scene.[5] As recently as 1995, Paolo Boero, of the University of Genoa, insightfully stated at a meeting of the Algebra Working Group of PME, that, through research, we had accumulated a large amount of sound knowledge on how people learn—or not—algebra, and on teaching strategies and approaches related to algebra education, and

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[2]The*Journal for mathematics teacher education* (Kluwer), whose chief-editor is Barbara Jaworski.

[3] http://www-personal.umich.edu/~dball/icmistudy15.html

[4] In a sense, this is a wishful statement. There are systems in which a given amount of mathematical 'training' at any university degree qualifies for teaching at school level, and in other systems, if the situation is not quite like that, there are serious discussions about adopting such a system; for instance, there has been a debate, in the US, about dropping 'pedagogical' courses as a requirement for certification, based on the idea that too many (specific subject matter) 'talented' people are being put off teaching because they do not wish to take those 'boring' courses, and end up in other professions. (from Jerry Becker's e-list)

[5]As one can see from examining the main journals of the field (*Educational Studies in Mathematics*, *Recherche en didactique des mathématiques* e *Journal for research ion mathematics education*, and from examining the proceedings of the annual PME meetings (International Study Group on the Psychology of Mathematics Education).

that, perhaps, it was time for us to begin thinking of ways of telling *teachers* about all that. And this is at least one step back in relation to stating that we should engage in developing ways to help teachers to be able to use all that knowledge developed and eventually made available to them (ways 'to prepare teachers').

It is true that in the field of ('pure') Education, the research tradition relating to teacher education goes further back in time (late 70's and early 80's), but, I will argue, that research could not, and cannot solve, by itself, the problems raised in relation to the professional development of mathematics teachers.

The existence of two separate and well established fields, Education and Mathematics, has, for a long time, offered support to the understanding that mathematics teachers must first properly learn the mathematics and then properly learn good ways of teaching it; this is clearly visible in the structuring of mathematics teacher education courses as 'a degree in mathematics' plus 'pedagogical complementation' ('3+1'), as found in so many countries. The support has been usually offered in the form of specific content-driven curricula and specialists that are able to teach within those curricula. The development and slow consolidation of specific educational areas—language education certainly came first, and mathematics education is, in size, reach and tradition, only behind language education—, initially did not much to challenge directly the 3+1 understanding.

More recently, however, the idea that teachers must learn 'the mathematics' *at the same time as they learn how to teach it*, has been put forward by a number of people in the mathematics education community (Cooney at al., 1996, is a good example of this), but other questions or headlines have also been brought forward. Some mathematics teacher educators began to ask questions about 'knowing and learning mathematics for teaching' (MSEB, 2001), 'what mathematics do mathematics teachers need' and 'what kind of mathematical experiences are adequate for the professional development of mathematics teachers' (for instance, in both cases, as in the work of Deborah Ball, but also in the considerations of Barbara Jaworski).

That line of questioning may be seen as simply furthering the cause of 'the mathematics simultaneously together with a pedagogy', but I think it does much more, that it actually challenges the very idea that the mathematics of the mathematician is suitable to promote, in itself, adequate professional development of mathematics teachers. And to say it is not suitable begs the question of what the mathematics of the mathematics teacher is, so it can become part of our efforts in teacher education.

'What is the mathematics of the mathematics teacher' is the central question I want to address in this paper, but in order to do that I will also have to touch upon what 'the mathematics of the mathematician' is. In both cases I will use a set of ideas related to what I call *meaning production processes*.

It might seem odd to characterise any 'mathematics' in terms of meaning production processes, and not in terms of, say, content (eg, definitions and theorems) and methods for establishing truths. My point, here is that, while for the mathematician—or, perhaps more precisely, for the philosopher of mathematics—that is a problem of capturing the 'essence' of something *already in place* and *well established as part—maybe central—of a social practice*, for the mathematics teacher such an approach is insufficient, precisely because no matter how much the teacher wants his/her students to think in a given way or to understand a statement in a given way, s/he simply cannot anticipate what the students will make of it. My characterisation of the mathematics of the mathematics teacher, then, is not primarily directed towards what the teacher him/herself thinks about or of mathematics, but rather towards what kind of *things*—to leave it less technical at this point—the teacher can 'see' as s/he *reads* students engaged in a mathematical activity, and this will take place as meaning production is happen*ing*, most of the time in situations of interaction.[6] Paraphrasing the pop song, I expect my characterisation to foster an interest on *humans being*, not on *human beings*, and to offer tools that can make this interest operational and useful in the classroom.

What I have said in the last paragraph has its root in the fact that a crucial aspect of the professional activity of a teacher—probably the most crucial aspect of it—is to make decisions and to take actions related to the mathematical education of his/her students[7], based on what s/he wants to achieve, but based also—and this is the key aspect I want to examine—on what is happening in the classroom.[8] To take the latter into consideration, it is not sufficient, I will argue here, to take *prima facie* what the students do or say; teachers have, as I said, to *read* what students are *actually* saying or doing.[9]

The way I will proceed from here is this. First I will introduce a set of notions, which are part of the Model of Semantic Fields, a theoretical model developed to characterise meaning production processes (Lins, 2001); those notions will offer the support for the arguments in the rest of the paper. Then, I will discuss a few exemplary examples of situations that a mathematics teacher might face in her/his professional practice, and then I will offer my characterisation of the mathematics of the mathematics teacher and will discuss some of its implications. Only then I will offer a characterisation for the mathematics of the mathematician, brief but sufficient for my purposes; the reason for choosing this order is that from the point of view of meaning production processes, the mathematics of the mathematician (MM, from now on) is a proper part of the mathematics of the mathematics teacher (MMT, from now on). Finally, I will draft a discussion of some consequences of all this to mathematics teacher education.

## THE UNDERLYING THEORETICAL FRAMEWORK: THE MODEL OF SEMANTIC FIELDS (MSF)

The central notions of the MSF are *object*, *meaning* and *knowledge*.

An *object* is, in the MSF, anything a person is talking about, be it 'concrete'—for instance, a chair in front of me—or 'symbolical'—for instance, letters in a piece of paper. *Meanings* are, in the MSF, what a person *actually* says of an object in a given situation

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[6]I make reference to mathematical activities, but with this I only want to characterise a situation that is familiar to mathematics educators, a situation in which there is something going on that we would agree it has to do with mathematics.

[7] See, for instance, Perrenoud (1999)..

[8]Just to clarify: I am not saying here that what has happened in the past, in those students lives—including inside other classrooms and ouside schools—does not have any sort of influence in what is happening at any given time, in a specific classroom. What I *do* want to say, though, is that tracing back what is happen*ing* to its 'roots' is a lost cause, and that the teacher would do better dealing with what is *present*. A long argument could be presented here in defence of this point of view, dealing with the problems of interpretation (in history, discourse analysis, psychoanalysis and many other fields); one could discuss, in a more general way, theories of the subject; naturally I will not do this here.

[9]Just to illustrate this point with a very common situation, one can simply recall that too often stdents who are asked 'why did you do it this way?', react exactly as if the teacher had said 'this is wrong'; one reason for this could be that also too often, if the answer or statement is 'right', that is, if it is something we ourselves would say, we are not quite interested in 'explanations'. This is similar to why children learn to say 'no' before they learn to say 'yes'.

(within an activity); it is not everything s/he *could* eventually say about that thing. And *knowledge* is, in the MSF, a statement-belief, something that a person actually states and in which s/he believes, *together* with the justification that person has for believing in that statement and for enunciating it (Lins, 2001).

Meaning production and knowledge production always happen together, and objects are constituted through meaning production.

Naturally, a number of points can be raised in relation to those characterisations, but I will direct my attention to only two of them: what is to believe and why to include the justification as part of the *knowledge*.[10]

First, what is to believe. I think a pragmatical answer is sufficient for my purposes: I will say a person believes in a given statement if s/he acts in agreement with it. For instance, if I say that I believe that people cannot see through brick walls and I need to check whether or not I left my keys on an adjacent room, I should not do this by looking in the direction of the wall common to both rooms.

The second point is the understanding of *knowledge* I propose. Traditional approaches to this question suggest that the nature of knowledge is that of a proposition (in my formulation, the statement), and attach to the justification only the role of allowing others to verify whether or not the person has reached that proposition through 'acceptable' ways (see, for instance, Dancy, 1993).

The theoretical problems involved in this conception are well known (the reader can refer to Dancy, 1993 or Chisholm, 1989). I will not present those objections here, but will, instead, provide a couple of examples that—hopefully—will make my choice acceptable; they are, again, of a pragmatical nature.

Consider a 5 yrs-old child who says 'two plus three is the same as three plus two', arguing that if you show two fingers on the left hand and three fingers on the right hand and bring them together, it is the same as showing three fingers in the left and two on the right and bringing them together. And consider a mathematician that says 'two plus three is the same as three plus two because the addition of integers is commutative'. Same statement, different justifications, different knowledges.

Another one. Consider a student and a teacher that both say 'if we know that 3x+10=100, then we can say that 3x=90'. The student is thinking of a scale-balance, thinking '**take**-the-same-from-both-sides'; the teacher is thinking about a numerical situation, thinking '**subtract**-the-same-from-both-sides'. Pretty close. But give '3x+100=10' to that student and and to that teacher, and watch what happens.

My main point here is that, in line with the linguist George Lakoff's understanding, whatever is central in human cognition has to show itself, has to be visible, in everyday simple situations. If we are not inclined to say that the child and the mathematician, and the student and the teacher had produced the same *knowledge*, we'd better do something about our understanding of that notion. My option is to include the justification as a *constitutive* part of it.

Those three notions are rooted in several considerations of a theoretical nature—related to theories of knowledge—, but they provide, nevertheless, a powerful practical tool for the teacher who wants to read meaning production processes as they happen, 'on-the-fly', as well as a tool for analysing students' written work and for developing tasks and tests items. In this paper I will concentrate on the first of the three.

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[10]Other issues are: legitimacy and interlocutors (how does one 'decide' whether it is legitimate to state something in a given situation?), leading to author-text-reader and the idea of communication; what is learning and what is it that we learn, from the point of view of the MSF (and relations between this and the work of Vygotsky and his colleagues). And others.

As a tool, the MSF aims at enabling the teacher to produce non-deficit readings of what the students are saying and doing.[11] If the teacher is able to say to a student 'I think this and that is what you are talking about', and the student agrees, *then* the teacher can say 'Well, I am thinking of something different from you, and I would like you to take a look at how I am thinking, is that OK?' and *then* productive interaction can happen.

**WHAT'S GOING ON HERE? SOME EXEMPLARY EXAMPLES**

The following examples are aimed at paving the way for what I will say in the next section. Many other examples exist or could be created. The reason for choosing this approach is that the MSF allows that a few examplary examples, together with simple enough 'principles'—'reading the student', 'meaning', 'knowledge, 'object'—open up the possibility that one greatly improves his/her ability to read what students say or do. In this sense, the following exaples, are, I think, useful.

SITUATION 1 (extending an example given above)

The students are working on simple linear equations; they have already studied negative numbers and done well. The teacher writes the equation 3x+10=100 on the board:

T: So, we can the say that 3x=90, right?
SS: Right.
T: …and then we can say that x=30, isn't it?
SS: Yes, that's right.
T: So the solution to 3x+10=100 is x=30…

The lesson is reaching its end, and the teacher gives the students a few other equations to solve for the next day, among them 3x+100=10, an equation the teacher considers almost too easy for them. The next day, as she asks the students for the solutions, a big surprise: not one student had solved 3x+100=10. The puzzled teacher asks the students what happened, and they say 'but Miss Julia, that one **can't be**…'.

What could be happening here? The teacher had focused herself on a mathematical meaning for the equation—an equation is an expression involving numbers, unknowns (or variables) and an equality—, and on associated meanings—for instance, 'if you do the same to both sides…'. 'Doing something' is, for her, related to adding, subtracting, multiplying or dividing both sides by the same number. Having done that, she cannot *see* what 'that one **can't be**' is.

What she did not know was that the previous teacher, who had introduced the students to linear equations, did so using scale-balance situations, and had used those situations a lot.

What is one to expect? That for the students those linear equations can only be scale-balance situations, *if anything*. Not knowing this, and not aware of the need to read what the students were actually saying or doing, as the students said 'remove' or 'take the same from both sides' she heard '**subtract** the same from both sides', and if they were saying 'subtract', why 3x+100=10 **can't be**?

[11]In fact, the model does not make a difference between saying and doing. Either saying is understood as doing something, or doing is understood as an enunciative act. And doing includes, for instance, gestures, arrangements or manipulation of physical objets, drawings and diagrams of all sorts.

The answer to this 'puzzle' should be clear by now: it is simply *impossible* to have a scale balanced by putting three equal things and, say, a 100 g weight on one side, and only a 10 g weight on the other one. That simply **can't be**, and this implies that students will do nothing about that problem, because if it is not a scale-balance it is nothing or, at best, it is a string of numbers, letter and other signs, and operations like 'remove…' do not apply to it.

The key point in this is that the teacher could not produce a non-deficit reading of the students' statement: mathematical meanings and right-wrong dominated her thinking, and the situation became paradoxical.

Even though the students were probably calling '3x+10=100' by the name of 'equation'—in agreement with the teacher's naming of it, the meanings they produced for that thing were not the same as the teacher's. Removing weights from both sides is not the same as subtracting the same, even though subtraction can be used to *work out* how the scale-balance will be after the removal. Similarly with 'sharing' or 'partitioning' in relation to 'dividing' (as in the arithmetical operation).[12] Because the meanings produced by students and the teacher were different to that extent, I will say the objects they were talking about were different and, naturally, the knowledge they were producing.

And we do not have to confine ourselves to those two 'kinds' of meaning: anyone can think of '3x+10=100' as a function machine situation, a whole-part situation or as a template against which to test candidates for getting a real numerical equality. In each case there would be different meanings, different knowledge, different objects.

SITUATION 2

A puzzled student-teacher comes to see me, with several pages of her students' work.[13] She had read numbers aloud and the students had to write them down with digits. A few examples of what they did:

| What she said | What they wrote |
|---|---|
| one thousand, two hundred and thirty-and-five[14] | 1000200305 |
| two thousand, six hundred and nineteen | 200060019 |
| three thousand and twelve | 300012 |

As I said, she brought *several* pages, all full of work like that, and she said she was absolutely lost, that she had *no idea* of what was going on. As soon as I pointed out to her that maybe they were writing *words with digits*, she *saw* what had happened. Her face lit

[12]I once had a very interesting conversation with Alan Bell, at the time my PhD supervisor. He argued that when a store-clerk gives you the right change by 'adding up', he is actually doing a subtraction. For instance, I have to pay $35 and give a $100 bill to the clerk. He gives me a $5 bill and says 'forty', gives me a $10 bill and says 'fifty', and finally gives me a $50 bill and says 'a hundred'. I argued that this and doing a subtraction were quite different things, as, unless the clerk wants to pay attention on how much he returned, he will not know, in the end, the change given (try yhis out in shops without those modern machines!). And how can we call 'subtraction' an operation that in the end leaves without knowing 'the result of the subtraction'? Shouldn't we better call that a 'change giving' operation? The same argument applies to 'sharing' and 'division'.

[13]In Brazil it is not uncommon that student-teachers are already teaching on their own, in schools; this is a consequence of the huge deficit in the numbers of certified teachers.

[14]That is how it would be in Portuguese, instead of 'thirty-five'.

up like when we suddenly *see* the solution of a problem that was challenging us, and the solution now is *obvious*. She could not see the 'obvious' before.

The key point is that the teacher could not produce a non-deficit reading of the students' statement: mathematical meanings and right-wrong dominated her thinking, and the situation became puzzling.

We may speculate that the objects they were working with were words, not numbers, something quite reasonable, given that the teacher was *speaking*. The digits, in that activity, had been given a meaning similar to that of letters and as much as in written languages there are different patterns for expressing similar sounds, those children 'wrote' thirty-and-five as 305 (see note 12) but nineteen as 19.[15] Because those meanings are not legitimate in the MM — if one ‘correctly’ reads back aloud the numbers the children wrote, they will not sound the same as what the teacher said, for instance — and because the teacher was initially not aware of the possibility of non-mathematical meanings, she could not ‘explain the error’, and all she could do was to insist with the children that they had made a mistake, and that the right way was such and such, so no productive interaction was likely to happen.

<u>SITUATION 3</u> (borrowed from Deborah Ball and Hyman Bass, and extended—I added student D)

Students are working on multiplication. One item is 47 x 25. Four solutions:

| **student A** | **student B** | **student C** | **student D** |
|---|---|---|---|
| 47<br>x25<br>235<br>94+<br>1175 | 47<br>x25<br>175<br>100+<br>1175 | 47<br>x25<br>35+<br>200<br>140<br>800<br>1175 | 47<br>x25<br>340<br>125+<br>1175 |

What's going on in each case? Student A's solution should be easy to *see*. The readers are invited to work out student B's and student C's solutions, but I would like to call the attention of the reader to student D's solution.

Every single time I presented this situation, the only understanding the audiences produced was that student D had copied the *result* from a colleague. When I asked why, then, he had not also copied the middle numbers, they replied that he did not had time, the teacher had looked in his direction, etc.. Fair enough, that *is* a possibility.

But what if student D was *very* good at mental calculation, although the only thing he could *say* about those middle numbers was that they had to be there in the *presentation* of the solution, and also the '+' sign: that was the *meaning* student D produced for that 'diagram'. So, he did it mentally and filled in the middle numbers with whatever numbers that came to his mind. That also is a possibility, and one that comes from an attempted, fictional, non-deficit reading. How to decide what was actually going on? The only way here is *to ask* student D, he has to tell the teacher his *justification* (as in the MSF).

The key point, here, is that for this student, thinking in terms of the ‘looks of the presentation of the solution’ was a legitimate thing to do, and as he believed in it, he acted accordingly. Not aware of the need to accept also non-mathematical meanings, the

[15]19 is written, in Portuguese, 'dezenove', that is, 'dez e nove', literally 'ten and nine', but surprinsingly few people notice this unless someone points it out to them.

audiences opted to assume that the student had cheated — something that, unwanted as it may be, is acknowledged as legitimate in tests, from the point of view of students, at least.

SITUATION 4 Thales

I was teaching an undergraduate mathematics course and, as an introduction, had shown to them a situation Deborah Ball had created[16]:

A primary school teacher has taught her students a unit on ordering decimal numbers. She now wants to prepare a test, and has developed three sets of decimal numbers, but wants to include only one of the in the test, with the requirement that the students arrange the four decimal numbers in decreasing order:

| | | | | |
|---|---|---|---|---|
| a) | 0.15 | 1.7 | 2.71 | 32.1 |
| b) | 1.2 | 0.13 | 0.232 | 13.5 |
| c) | 9.08 | 0.75 | 3.72 | 0.068 |

The question is: should she prefer any of the three items to the others?

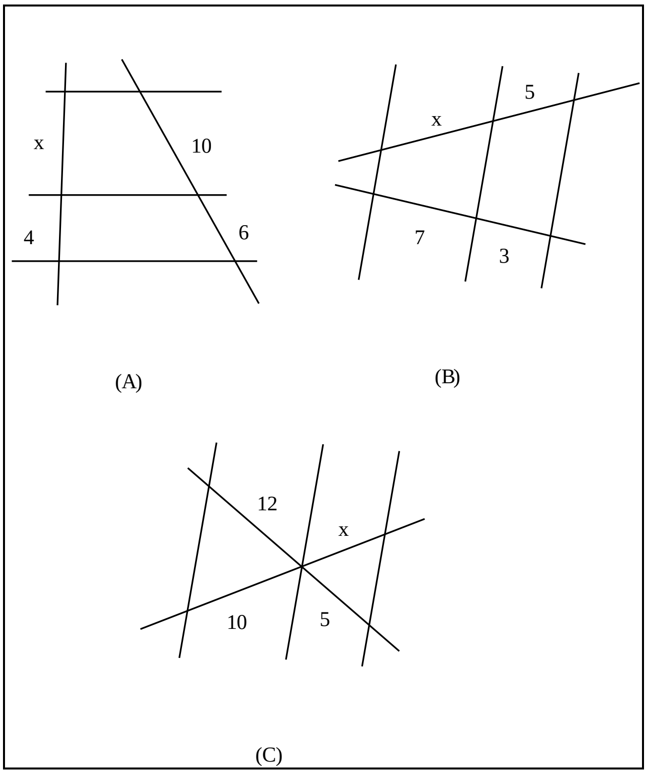
After presenting a number of people with this situation, she noticed that much more often than not, (school) mathematics teachers said they would choose item (b), while (university) mathematics professors said it did not make much difference. When I tried this in Brazil I got quite similar results. The reader may wish to take a minute to think about this: why did teachers choose item (b)?

The answer seems to be that teachers know that students quite often, when comparing decimal numbers, simply drop the decimal dot and compare the numbers resulting from the remaining digits. And only in item (b) this procedure will fail to produce a correct answer…!

The whole point is, and this is the reason I presented and discussed this situation with my students, that for the mathematics professors, the mathematicians, 'droping the decimal dot' is not a *legitimate* operation, or action, on decimal numbers. So, I gave this single example, pointed that out and said to them "The teacher has to read his or her students, and reading is this, it is trying to produce a non-deficit understanding of the students' thinking". Maybe 15 minutes elapsed.

At the end of the following lesson one of my students came to see me; he was already teaching at a school (as in situation 2). He said that because of what I had said in the previous lesson, he had been able to solve a mistery that challenged him for some time. He had given a test to his students, which included the following item, related to Thales theorem (figure below):

[16] I have taken the idea, but not used any numbers Deborah Ball had actually picked.




He said that almost all of his students had got (A) and (B) right, but only very few got (C) right, and he could not find any visible reason for that, until after I had said what I said, when he went back to the test sheets and *saw it immediately*.

Students who got (A) right used the scheme $\frac{x}{4} = \frac{10}{6}$, while students who got (B) right used $\frac{x}{7} = \frac{5}{3}$. My student said that at the time of marking the tests he tought the solutions to (B) to be unexpcted, as in the classroom he had always composed the ratios with the quantities corresponding to segments lying on a same line, so in (B) he expected them to write, for instance, $\frac{x}{5} = \frac{7}{3}$, as he had never showed to them that the two schemes were equivalent.

As he took aboard what I had said, and started to think about non-mathematical meanings, he immediately realised that the meanings that had guided their actions-solutions was not directly related to Thales' theorem. Apparently they knew that they had to set up a proportion and then to solve it for x, but the choice of what went where in the proportion was guided by the visual disposition of the elements on each subitem. In (A) the x is *above* the 4, so in the first ratio the x should also go above the 4, and similarly with the second ratio. In (B) the x is above the 7, so in the first ratio it also goes above the 7. And so on.

And then the solution to the mistery slowly emerged from within the mists. As he examined his students' 'wrong' solutions to (C), he realised that in all cases they had used the scheme $\frac{12}{10} = \frac{x}{5}$, in complete coherence with what they had done in (A) and (B).

There are two key points I want to emphasise.

First, that as I had said above, the principles and tools offered by the MSF, together with a few exemplary examples—in this case a single one—can make a huge difference in teachers' capacity to read what his or her students say or do, but also that it worked that well for a teacher with little experience, and that each time a new instance of that kind reading will happen, much more important than the teacher's repertoire growing, his feeling for that kind of situation and and that kind of process will be more refined.

Second, that as much as in Deborah Ball's example, that I had presented to my students, in the case of this teacher it was the introduction of the *theoretical* notion of non-mathematical meanings that solved the mistery.

SOME DISCUSSION

Those examples could be multiplied many times, either from actual classroom situations or from fictional ones. And they would come from primary school classrooms to university level mathematics courses (Lins et al., 2002), and would be related to any situation that a mathematics teacher would call 'mathematical'—pure or apllied, word problems or theorems, definitions or models. Also, it does not matter if we are dealing with situations that 'look right', as in situation 4, (a) to (c), situations that 'look wrong', that is, we clearly see that the answer is wrong, or if we are preparing ourselves to teach, as in Deborah Ball's decimals choice situation.

From a purely practical, classroom, point of view, someone might say that what is in the examples above could be elicited and discussed without any reference to the MSF and to the notions it proposes, and I would have to agree with him or her to that extent.

But the introduction of the MSF has a key impact in two aspects related to those practical situations.

First, the notions proposed by the MSF offer a general principle—*read the students*—and tools for actually exercising that principle—*what are the objects the students are thinking about/with? what are the meanings they are producing for those objects*. Those two questions have a degree of generality that makes them flexible enough to be equally useful to beginning teachers and to experienced teachers. There should be no doubt that experience in doing that reading makes one more able to *see*, but that is not so because experience may give a teacher a bigger repertoire of typical situations: what one develops is an actual *intuition* and a *habit* of doing it, to the extent that at some point producing non-deficit readings becomes automatic, and that is what makes the introduction of those theoretical constructs relevant in teacher education. I think this is a significant aspect of the professional development of the teacher, as it will greatly improve the possibilities of productive interaction, at the same time it definitively puts the students at the centre of the teacher's voluntary attention. And in saying this, I want to point out that it is the support of a theoretical model that allows me to propose and discuss notions like that and to examine meaning production processes in a sufficiently detailed way, so to make this kind of mathematical experience a rich environment for professional development.

Japanese teachers refer to a practice called 'hyouka', which means 'looking at' the student aiming at *seeing* him or her with the intention of doing something to help the student to learn this or that, and then doing something. But what is the teacher to look for as he 'looks at' the student? The MSF offers an answer: objects and meanings as proposed in the model.

The second aspect in which the introduction of the MSF has an impact is the understanding one has of the very practice of the mathematics teacher, an aspect that has, naturally, quite deep implications on what one considers adequate for mathematics teacher education.

It should be fairly obvious to anyone who is, or has been, a mathematics teacher, that in the classroom there is not 'the mathematics' on one side and 'the pedagogy' on the other: as the teacher makes decisions and take actions, considerations of all sorts are involved. Even teachers who simply lecture 'the mathematics' take that decision based on beliefs about how learning happens, what 'the mathematics' is, or what 'thinking mathematically' is, even if those considerations remain at an deological level.

But if things are not separate, why should one conceive—as too many people still do—the professional education and development of mathematics teachers as if they were? As I mentioned on the Introduction, colleagues like Tom Cooney have dealt with this problem by proposing development experiences (courses, for instance) in which the mathematical discussion goes hand in hand with the pedagogical discussion. As I said before. And, as I said before, I think we should go one step further.

The introduction of the MSF brings about the fact that, from its point of view, the mathematics of a mathematics teacher is neither a subset of the content and methods of the mathematics of the mathematician—those parts which are relevant to school mathematics, Didactical Transposition (Chevallard) considered—, nor is it a sort of ethnomathematics of the teacher. It is directed to *processes* and *interaction*, and not to characterising what mathematics is—and from there defining right and wrong and what should be properly taught—or to enable the teacher to control what is right and wrong in what students are saying or doing—as it is the case with the model of Conceptual Fields, proposed by Gerard Vergnaud. And it is not intended to describe or prescribe what the teacher knows or should know of, thinks or should think about mathematics; instead, the MMT consists of an awareness of and a willingness to read meaning production processes.[17]

To put it in simpler words, the MSF is aimed at ‘understanding as knowing the students thinking *with the intention of interacting with them*’, and not at understanding as ‘being able to explain the errors *in order to correct them*’. That is, the MSF and the MMT has primarily to do with the *students*.

The stance taken by the MSF establishes creates some simmetry in the power relations in the classroom and in the whole learning process, and makes respect an intrinsic part of those classroom processes: as teacher I am in a position to say to my students 'I think I understand how you are thinking; I am thinking differently. Would you like to take a look at how I am thinking? This may help you to understand what I am trying to teach you', and this will not represent at all an attempt to 'erase' the students' other ways of thinking, but precisely at *expanding* their thinking possibilities.[18]

All that said, I can now move to the next section, where a characterisation of the mathematics of the mathematics teacher is proposed.

[17] To a great extent, this is a characterisation that does not depend on the ‘subject matter’ being or not ‘mathematics’. But as I shall clarify ahead, the fact that the ‘subject matter’ is mathematics, together with what the mathematics of the mathematician is, do give very peculiar tones to the MMT.

[18] Perhaps one could draw a paralell here, between my viewpoints and those defended by Paul Feyerabend, in his *Against Method*.

## THE MATHEMATICS OF THE MATHEMATICS TEACHER

> *The mathematics of the mathematics teacher is characterised by its acceptance of non-mathematical meanings for things that might be otherwise called 'mathematics'.*

In some cases those non-mathematical meanings are quite well-known and accepted in schools, for instance 'equations are scale-balances', which are actually used as resources to (supposedly) facilitate learning.[19,20] But there are many instances in which the non-mathematical meanings are only understood or explained as errors, as in Situation 2 and Situation 3; these are critical situations because the statements of student and teacher do not agree, but as Situation 1 shows, an agreement in the statements can be equally problematic as the process unfolds.

The MMT accepts non-mathematical meanings as *legitimate for the student*, and *for that reason*, legitimate in the classroom; that is why the teacher has to engage in trying to produce a non-deficit reading of what the students' are saying or doing. The spatial disposition of elements in a drawing or diagram count; decimal dots can be disregarded; 'space' is a natural notion—and a single one, not the many 'vector space', 'metric space', 'topological space' and so on—, and 'plane' a naturalised one, not the one defined in the mathematics of the mathematician (see Lins et al., 2003).

Let's emphasise this here: the MMT, as I understand it, is not charcterised on the basis of content or ways of establishing truths. I am saying this in order to mark my intention as clearly as possible. If a mathematics teacher must or not know this or that part of statistics, algebra or any other content of mathematics, is, in itself, a question to be decided on the basis of curricular demands, a policy matter, and on the basis of which of those contents can be suitably used to promote the discussion of meaning production processes, difference and reading the students (as in Lins, 2002, 2004), precisely because, independently of any content, the mathematics teacher has to read his or her students in order to *see* what is happening and become able to promote interaction.

The issue mentioned just above— secondary from the point of view of this paper, but equally important—is that of the mathematical education of mathematics teachers, initial and continued development. What kind of experiences can provide the teacher with an awareness of difference in meaning production and promote the development of the ability to read meaning production processes? In Brazil, prospective mathematics teachers take courses on Calculus, Abstract Algebra, Linear Algebra, Analysis, Metric Spces and Topology, and so on, almost always from the perspective of the mathematics of the mathematician. Why?

Current discourse makes reference to 'content to be taught in school'—as defined by local demands—and to 'foundations'. When I consider the first, I have to ask myself why not requiring them to take mostly courses that actually deal *directly* with school

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19 Elsewhere I have argued that the supposed facilitation may end up in tragedy: the teacher moves away from one 'metaphor' to another, but forgets to tell the students of the move. As in Situation 1, the possibility of interaction is seriously harmed. Other examples are abundant: begin with 'decimal numbers as money' and then multiply $3.20 by $4.00, getting the result 12.80 square $; begin with 'fractions as cake slices' and then multiply cake slices. And so on.

20 Much of that acceptance relies on the idea that the mathematical meaning of 'equation' is, *in fact*, the *essence* of what is being said in relation to the scale balance. See, for instance, my critique of this in Lins (1992)

mathematics. When I consider the second, I have to ask myself whether or not Euler — who knew nothing of ε‘s and δ‘s, groups and rings and fields, vector spaces, non-euclidean geometries —would be *mathematically* qualified to be a school teacher nowadays.

This is not to say that courses structured around the mathematics of the mathematician are of no use. They might be, particularly as means to discuss difference and meaning production (mathematical meanings/non-mathematical meanings); I will state, however without providing here any further support to this statement, that the *categories* that structure the mathematics of the mathematician (as seen on the names and sillabi of mathematics courses) generally offer much less oportunities to provide prospective teachers with the experiences they need, than other categories; in the last section I will comment a little more on this .

Summarising. The MMT is to be characterised in terms of meaning production processes and legitimate modes of meaning production, not in terms of content. The central aim is to broaden the scope of meanings acceptable, *readable* — that is, the centre is in the *reading* capacity of the teacher, which is directed *towards the students* —, not to narrow the content—that is, the centre is not in the reproductory capacity of the teacher. And, in didactical terms, we must always bear in mind that the student has the *right* to know when meaning production by the teacher changes.

That leads us to the next section.

## THE MATHEMATICS OF THE MATHEMATICIAN[21]

The most distinctive feature of the MM is that as soon as things are defined, that is what they *are and will be until further notice*. Or, just to create an image: there is no other area of the human endeavour in which its practitioners have so much control over what the things they deal with are or are not, as the mathematics of the mathematician.

The MM is a deductive science, but without the distinctive feature I have pointed out, the deductive chains would not work the way they do in the MM, as R. D. Laing exemplifies in his book *Knots*.

The logic through which one proceeds in the establishment of truths might vary—classic, para-consistent or fuzzy, for instance—, but this simply creates new fields, not necessarily conflicts. The central feature of the MM remains untouched: things are *defined* And defining, saying ‘this is this and that is that’ amounts, in fact, to meaning production as proposed by the MSF. In other words, the MM circumscribes what things are, by opting for a definitional mode of meaning production. Moreover, definitions in the MM are used to *constitute* objects, not to *describe* objects.

To state it briefly, what the MM is today is the result of a process of a kind of ‘cleansing’ that began roughly at the first half of the 19th century and was somewhat settled by the 1930’s with the Bourbaki initiative. Intuitions dependent on the ‘phisycal world’ were banned, in order to avoid ‘mistakes’ generated by false ‘perceptions’. From Hamilton on, integers were no more than constructions, creations based on other soundly created things, and not debatable things.[22] And Cantor’s administration of an infinity bigger than another definitely set the character of the Mathematician’s Garden (Lins, 2004).

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[21]A disclaimer: this section is not intended *in the least* to be a fully fledged discussion of the MM, as it only discusses aspects of it which I think are relevant to better situate the discussion of the previous section; elsewhere the MM will be discussed in depth, in terms of meaning production.

[22]In the 17th century, Leibniz was challenged by the theologist Arnaud, who argued that this thing of negative numbers was an absurd, once -1:1::1:-1 implies a situation in which a

This is my main point here: all that process was directed towards *restricting* the meanings that were legitimate to be produced for certain things within mathematics, named ‘real numbers’, ‘limit’, ‘infinity’, and so on, and, by doing that, restricting the authority to talk about 'mathematics' to those who stuck to the definitional mode of meaning production — abstract, non-'concrete'. It is fine, in the MM, to define 'surreal numbers' and develop a whole theory about them, even though they look like numbers as much as bees and elephants resemble one another. But Surreal Numbers are *legitimate* in the MM regardless of the fact that they generated interest in studying them, they are legitimte because they conform to the legitimate modes of meaning production in the MM.

Most of the *facts* of mathematics—its theorems, propositions proven to be true—were preserved in that process, although what they referred to had substantially changed. It is quite likely that Euler knew that the sum function of two functions continuous at an interval is continuous, although ‘function’ and ‘continuous’ were, for him, something quite different from what they are today.

Contentwise—in the sense of the statements themselves—things were preserved. Meaningwise, they were not. And, similarly, even if statementwise things are preserved—from the teachers point of view— as students speak/do, meaningwise they might not be. And even more so, there are students' statements that don’t even agree with the teacher’s, and still they have to be understood *in their own terms*.

*Culturally*, everything the mathematician says about mathematics is right — mistkes discounted, of course[23]; mathematicians are precisely the people who have the authority to say what is right and what is wrong in mathematics, and this is a result of the process of professionalisation related to the 'cleansing' I mentioned.[24] But students—and most people in the world, for that matter—know nothing of that, so they may have other views; they might want to say things the mathematician wouldn’t. And the mathematics teacher has to deal with that.

It is in this precise sense, that the MMT is ‘bigger’ than the MM, that the former encompasses the latter: everything the mathematician can say about ‘mathematical matters’, the teacher—and, more importantly, the student—could eventually say, but not the other way round.

This shows that meaningwise the MMT encompasses the MM, that is: in a sense, Mathematics is a subarea of Mathematics Education.

## FINAL REMARKS AND DIRECTIONS

In this paper I attempted to provide an understanding of the professional practice of the mathematics teacher that departs from the traditional understanding. Instead of being

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smaller is to a greater as a greater is to a smaller. Leibniz answered to Arnaud's objection, saying that Arnaud was right to that extent, but that nevertheless he (Leibniz) would go on using negative numbers, because *they worked*. (see Lins, 1992)

[23] One notable case comes to mind, that of ‘Cauchy’s mistake’. In Baldino et al. (2001), however, the authors argue that Cauchy did not make a ‘mistake’, instead he ‘thought differently’ from what is today the current view.

[24]Referring back to footnote 17, it was not a problem that Arnaud had said that, as anyone can say whatever s/he wants. But it is quite intriguing that Leibniz took time to answer to Arnaud's objection. Would that happen today, even if related to a more 'sophisticated' mathematical subject, the mathematician's answer—if there would be one at all—would probably resemble Louis Armstrong's reply to a woman who had asked him "Mr. Armstrong, what is Jazz?": "As long as I have to explain it to you, it is useless doing it…". (Se non é vero, é bene trovato)

the person who will try to teach his or her students some part of mathematics—applied, in models, pure or whatever—, then check whether the students are responding 'well' to the teaching and, if necessary tries to correct what is wrong, I propose that the centre of the teacher's practice is to read what the students are saying/doing so interaction can happen. And interaction is to be understood, here, as 'sharing modes of meaning production', so teaching becomes a matter of cultural immersion.

But in order to do that, the mathematics teacher has to know a mathematics that neither is the mathematics of the mathematician, nor it can ever be described in terms of content, as we are dealing with processes. For that reason I characterised the MMT in terms of meaning production processes, based on the MSF.

Apart from the consequences I have already pointed out, there are others worth discussing, but for the sake of the length of the paper I will consider only one of them.

We must seriously consider the question 'what kind of mathematical experiences are adequate for the professional development of mathematics teachers?', already mentioned in this paper. More specifically, we need to investigate to what extent the courses we offer today to mathematics teachers—pre- and in-service—can foster the development of an awareness of difference, in the sense of meaning production. Can (pure) mathematics courses provide that? Can pedagogy courses—general pedagogy or methods courses—provide that?

Almost a century ago, Felix Klein wrote, in the Introduction to his *Elementary mathematics from an advanced standpoint*,

> […] *For a long time* […] *university men were concerned exclusively with their sciences, without giving a thought to the needs of schools, without even caring to establhish a connection with school mathematics. What was the result of this practice? The young university student found himself, at the outset, confronted with problems which did not suggest, in any particular, the things with which he had been concerned at school. Naturally he forgot these things quickly and thoroughly. When, after finishing his course of study, he became a teacher, he suddenly found himself expected to teach the traditional elementary mathematics in the old pedantic way; and, since he was scarcely able, unaided, to discern any connection between this task and his university mathematics, he soon fell in with the time honored way of teaching, and his university studies remained only a more or less pleasant memory which had no influence upon his teaching.*

I am tempted to say that the situation has barely changed in many places, if not most — as s/he enters the teaching profession, after the university studies, the teacher will almost always take his or her own school experience as the reference for his or her teaching — but I will refrain from assuming this in full, because the report of a survey by Wilson, Floden and Ferrini-Mundy (2001) shows that there is no sound corpus of research examining the impact of content courses on the actual practice of future mathematics teachers; even adequate procedures for such studies are yet to be developed. The strong suggestion is that we need carefully designed and conducted studies to examine this situation, and we need, then, to develop sets of categories that can actually have an impact, through mathematics teacher education, in the actual practice of teachers.

The characterisation of the MMT I have offered moves, I think, the reference post for mathematics teacher education and development: instead of 'mathematics' plus 'pedagogy', the reference post must be in 'mathematics education'. Mathematics teachers need courses on mathematics education, courses that can emulate actual classroom environments and the thinking that goes on there and the kinds of demands s/he is subjected to in that professional practice (Lins et al., 2002). And, repeating what I have said before, it is central that in mathematics educations courses, mathematics teachers experience and discuss meaning production processes, non-mathematical and mathematical meanings and the differences between them.

This seems at least a quite promising way to help them to ‘learn’ the mathematics of the mathematics teacher, a key part of the task of promoting *education through mathematics*.

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